Extra Credit problems

Logic and Algebra

1. You have two American coins that total 30 cents. One of the coins is not a quarter. What are the two coins?
   Quarter and nickel

2. Some months have 31 days; how many have 28?
   All of them

3. Why can’t a man living in the USA be buried in Canada?
   He is alive!

4. Before Mount Everest was discovered, what was the highest mountain on Earth?
   Mount Everest

5. How many animals of each sex did Moses take on the ark?
   Wrong guy: Noah took animals on the ark! Not Moses!

6. A plane full of English tourists travels from Holland to Spain. It crashes in France. Where should the survivors be buried?
   Survivors shouldn’t be buried!

7. If you drive a bus with 42 people on board from Gary to Chicago and drop off 3 people at each of the six stops and pick up 4 people at half the stops, when you arrive at Chicago 2 hours later what is the driver’s name?
   Whatever your name is: You drive the bus!

8. Do they have a 4th of July in England?
   Yes - they just aren’t as happy about it!

9. If I took 12 apples from a pile of 23 apples, how many apples would I have?
   12 apples!

10. Is it legal for a man in California to marry his widow’s sister? Why?
    He is dead!

11. A woman gives a beggar 50 cents; the woman is the beggar’s sister, but the beggar is not the woman’s brother. How is that possible? Explain this.
    The beggar is a female beggar! They are sisters.

12. If you have only one match and you walked into a room where there was an oil burner, a kerosene lamp, and a wood burning stove, what would you light first?
    Most likely - the **match**!

13. Divide 30 by 1/2 and add 10. What is the answer?
    $30 \div \frac{1}{2} + 10 = 70$

14. Why are 2013 dollar bills worth more than 2012 dollar bills?
    Because 2013 is more than 2012!

15. The 22nd and 24th presidents of the United States had the same mother and the same father, but were not brothers. How could this be so?
    **Grover Cleveland**: Cleveland was the only President to serve two nonconsecutive terms. He was thus America’s 22nd and 24th President.
16. Becky’s mother has three daughters. She named the first daughter Penny and her second daughter Nichole. What did she name the third daughter?

Becky

17. If it takes $6 \frac{1}{2}$ minutes to boil an egg, how long does it take to boil five eggs?

$\frac{1}{2}$ minutes (in a large pot)

18. A farmer had 17 sheep. All but 9 died. How many did he have left?

9

19. A book costs $1 plus half its cost. How much does the book cost?

\[ x = \text{cost of the book} \]
\[ x = 1 + \frac{1}{2}x, \text{ so } x = 2 \]

20. It takes you 30 seconds to walk from the first (ground) floor of a building to the third floor. How long will it take to walk from the first floor to the sixth floor (at the same pace, assuming that all floors have the same height)?

From first floor to third floor: 2 flights of stairs -> 15 seconds each! = 30 seconds

From first floor to sixth floor: 5 flights of stairs -> 15 seconds each! = 75 seconds

21. If four cows produce 4 cans of milk in 4 days, how many days does it take eight cows to produce 8 cans of milk?

SOLUTION So: 1 cow produces 1 can in 4 days., eight cows will produce 8 cans in 4 days.

22. If a farmer has 5 hay stacks in one field and 4 hay stacks in the other field, how many hay stacks would he have if he combined them all in the centre field?

1 haystack in the middle of the centre field!

23. At midnight it is raining hard. How probable is it that it will be sunny in 72 hours time?

Midnight is in the middle of the night: 72 hours later it is still midnight, so it can’t be sunny.

24. If it takes a man one hour to dig a hole two meters long, two meters wide, and two meters deep, how long would it take the same man to dig a hole four meters long, four meters wide, and four meters deep, assuming he digs at the same rate of speed.

SOLUTION 8 hours.

25. What day would tomorrow be if yesterday were five days before the day after Sunday’s tomorrow?

SOLUTION

Sunday’s tomorrow = Monday

day after Sunday’s tomorrow = Tuesday

Yesterday is Five days before = Thursday

Today = Friday, so Tomorrow = Saturday

26. What is wrong with the following argument?

(a) 12 eggs = 1 dozen (multiply both sides by 2)
(b) 24 eggs = 2 dozen (divide both sides by 4)
(c) 6 eggs = $\frac{1}{4}$ dozen (multiply equation b with equation c)
(d) 144 eggs = 1 dozen (12 eggs = 1 dozen)
(e) 12 dozen = 1 dozen

Solution: (d) (This should be dozen $\times$ dozen = dozen$^2$ = 144 eggs)
27. What is half of 2+2?

\[
\frac{1}{2} (2) + 2 = 3
\]

28. A snail climbs up a 30 feet high wall. During the day he climbs 5 feet, but every night he slides 4 feet down. After how many days will he reach the top of the wall?

SOLUTION

<table>
<thead>
<tr>
<th>Day</th>
<th>Up during day (feet)</th>
<th>Slide to, during night</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>24</td>
<td>28</td>
<td>24</td>
</tr>
<tr>
<td>25</td>
<td>29</td>
<td>25</td>
</tr>
<tr>
<td>26</td>
<td>30: GETS OUT</td>
<td>Doesn’t slide anymore!</td>
</tr>
</tbody>
</table>

29. The figure below depicts a bull made of matches looking to the left.

![Bull Diagram](image)

By moving only two matches can you make bull look in the other direction? (No doubling, or breaking matches allowed, all matches must be used.)

30. The figure below depicts a martini glass made of matches containing an olive

![Martini Glass Diagram](image)

By moving only two matches and without touching the olive itself, can you get the olive outside of the glass? No doubling, or breaking matches allowed, all matches must be used.)

31. \[VII = IIV = I\]

By moving just one match, can you make the equation true?

32. The figure shows five squares of equal size made from 16 matches. By moving only two matches, create four squares of equal size. All 16 matches have to be used. Doubling, or breaking matches is not allowed.
33. Rearrange 6 matches so that they make 4 equilateral triangles (and nothing else) of the same size? (no breaking or doubling – all matches must be used).

Think in three dimensions!

34. The story of Jim's life

At age 7 he began his schooling, which lasted for a quarter of his life. After finishing his education he went to India, living for 5 years with a guru, searching his inner-self. After returning to this country, he spent half of his life working as a doctor. He spent the last eighth of his life in a Florida retirement community. How old is Jim?

\[ \frac{7}{4} + \frac{x}{5} + \frac{x}{2} + \frac{x}{8} = x \]

Solution is : \( x = 96 \)

35. I think of a positive number, multiply it by 3, add ten, take away the number I first thought of, divide the answer by 2, and square the result. Then I subtract 25, and divide by the number I first thought of. My answer is 23. What number did I think of?

\[ x = \text{positive number} \]

multiply it by 3, add ten, take away the number I first thought of: \( 3x + 10 - x = 2x + 10 \)

divide the answer by 2, and square the result. \( \left( \frac{2x + 10}{2} \right)^2 = (x + 5)^2 = x^2 + 10x + 25 \)

subtract 25, and divide by the number I first thought: \( \frac{x^2 + 10x + 25 - 25}{x} = \frac{x^2 + 10x}{x} = x + 10 \)

My answer is 23: \( x + 10 = 23 \):

\[ x = 13 \]

36. Two boys on bicycles, 20 miles apart, began racing toward each other. The instant they started, a fly on the handle bar of one of the bikes started flying toward the other bike’s handle bar. As soon as it reached the other handle bar, it turned around and went to the other bike and so on until the bikes finally met. If each bike had a constant speed of 10 mph, and the fly was traveling at 15 mph constantly, how far did the fly travel?

Bikes travel at 10 mph, so after 1 hour they meet!

Fly flies at 15 mph for one hour, so it travels 15 miles.

37. Four different pumps are filling an aluminum pool, of dimensions 3 \times 3 \times 7 feet. Every hour the amount of the water in the pool doubles. The first pump pumps twice as fast as the second, and three times as fast as either third or fourth. The pumping begins on Tuesday at 8:00 a.m. and ends on Wednesday at 2:00 p.m. The attendant is reading a book called "I love Finite Math" next to the pool, turning pages at the pace of one page per minute. She is drinking iced tea chilled to a comfortable 55°F. The temperature of the air is 95°F and the temperature of the water is 80°F. At 4°C the density of the water is 1 g/cm³, while at 20°C the density of the water is 0.9982 g/cm³. At what time is pool exactly half-full?

Every hour the amount of the water in the pool doubles: Wednesday 1 p.m.
38. Boat is docked on the surface of the sea in the Boston Harbor. The tide begins at 7:00 a.m. rising the level of the sea at the rate of 1 feet per hour. A 29 foot ladder is hanging on the side of the boat. The steps are 1 foot apart, and the lowest step is one foot below the surface of the sea. When will the water reach the 15th step on the ladder?

Never : Boat rises with the tide!

39. The tanker “Wenwood” is docked in San Francisco. At 10:00 a.m. yesterday 17 water-line marks, each 6 inches from the next, were visible above the surface. The ship was being loaded so that it sank one foot into the water each hour. The tide was rising at a rate of 8 inches per hour until it reached its peak at 2 p.m., when it began to go down 7 inches every hour. At 3 p.m. the loading rate was decreased so that the ship sank at only 4 inches per hour. How many water-line marks were visible at 6 p.m., when loading stopped?

SOLUTION:

\[
egin{align*}
10am & : 17 \times 6 = 102 \text{ in above water} \\
loading(10 - 3 \text{ is 5 hours}) & : -12 \text{ in/hour} = -60 \text{ in down} \\
loading(3 - 6 \text{ is 3 hours}) & : -4 \text{ in/hour} = -12 \text{ in down} \\
TOTAL & : 102 - 60 - 12 = 30 \text{ in above water} \\
marks 6 \text{ in apart} & : \frac{30}{6} = 5 \text{ water marks visible}
\end{align*}
\]

40. Say that cucumbers are on sale, so you buy 100 pounds of them at your local market. The cucumbers are 99% water. Some days later, they dry out to 98% water. How much do they weigh now?

SOLUTION

Dry weight of the cucumbers is \( a \) pounds. Therefore, of the 100 pounds at the beginning, 99% is water, so only 1% is dry weight: \( a = 1 \) lb.

After a few days, 98% is water, so the dry weight is 2%. Dry weight is still the same, \( a = 1 \) lb. So, 2% of what number is 2 pounds?

\[
0.02 (x) = 1 : x = \frac{1}{0.02} = 50 \text{ pounds}
\]

The weight is now only 50 pounds!

41. Show that for every natural number \( n \in \mathbb{N} \), and for any \( n \) points in the \( xy \) plane, we can find a straight line which does NOT go through any of the \( n \) points, i.e., completely misses all of them.

SOLUTION:

Let the points be \(( x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\). Let \( x_{\min} = \min \{x_1, x_2, x_3, \ldots, x_n\} \) and \( y_{\min} = \min \{y_1, y_2, y_3, \ldots, y_n\} \). Then the point \((x_{\min} - 1, y_{\min} - 1)\) is outside (to the left and below) of all the points. Either of the lines \( x = x_{\min} - 1 \), \( y = y_{\min} - 1 \) or even \( (y - y_{\min}) = -(x - x_{\min}) \) will miss ALL of the \( n \) points.

42. Prove that \( \log_{10} 2 \) is irrational.

SOLUTION:

Suppose \( \log_{10} 2 \) is rational. Then there exist integers \( p \) and \( q \), such that \( \log_{10} 2 = \frac{p}{q} \).

\[
\log_{10} 2 = \frac{p}{q} \text{ so } 10^{\frac{p}{q}} = 2
\]

\[
10^p = 2^q
\]

\[
2^p5^p = 2^q
\]

\[
5^p = 2^{q-p}
\]

But any power of 2 (except \( 2^0 = 1 \)) has the last digit 2, 4, 8, 6, 2, 4, 8, 6, 2, ... and any power of 5 (except \( 5^0 = 1 \)) has the last digit 5, 0, 5, 0, 5, 0, ...

These two can therefore never be the same, unless \( p = 0 \) and \( q - p = 0 \), so \( p = 0 \) and \( q = p = 0 \), which is impossible. Therefore \( \log_{10} 2 \) can not be a rational number.
43. Solve the equation below for \( n = 1 \), for \( n = 2 \), for \( n = 3 \) and for \( n = 4 \). Then, solve the equation for a general natural number \( n \).

\[
\sum_{l=0}^{n} (-1)^l \binom{n}{l} \left( \sum_{k=0}^{l} \binom{l}{k} (x-1)^k \right) = (n-1)^n
\]

**SOLUTION:**

For \( n = 1 \) : \((1-x) = 0 \) : \( x = 1 \)
For \( n = 2 \) : \((1-x)^2 = 1 \) : Solution \( x = 0, x = 2 \)
For \( n = 3 \) : \((1-x)^3 = 2^3 \) : Solution \( x = -1 \)
For \( n = 4 \) : \((1-x)^4 = 3^4 \) : Solution \( x = -2, x = 4 \)

In general:

\[
\sum_{k=0}^{l} \binom{l}{k} (x-1)^k = \sum_{k=0}^{l} \binom{l}{k} (x-1)^{l-k} = ((x-1) + 1)^l = x^l
\]

Plug it in:

\[
\sum_{l=0}^{n} (-1)^l \binom{n}{l} \left( \sum_{k=0}^{l} \binom{l}{k} (x-1)^k \right) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} x^l = \sum_{l=0}^{n} \binom{n}{l} (-x)^{l} 1^{n-l} = (-x + 1)^n
\]

Therefore, the equation is the same as

\[
(1-x)^n = (n-1)^n
\]

For even \( n \) :

\( 1 - x = \pm (n - 1) \) : Solutions: \( x = 1 \pm (n - 1) \); so \( x = 2 - n \) or \( x = n \)

For odd \( n \) :

\( 1 - x = (n - 1) \) : Solutions: \( x = 1 - (n - 1) = 2 - n \)

44. Let \( a \) and \( b \) be the roots of the equation \( x^2 - mx + 2 = 0 \). Suppose that \( a + \frac{1}{b} \) and \( b + \frac{1}{a} \) are the roots of the equation \( x^2 - px + q = 0 \). What is \( q \) ?

(A) \( \frac{5}{2} \) \hspace{1cm} (B) \( \frac{7}{2} \) \hspace{1cm} (C) 4 \hspace{1cm} (D) \( \frac{9}{2} \) \hspace{1cm} (E) 8

\[
(x-a)(x-b) = x^2 - (a+b)x + ab : ab = 2, (a+b) = m
q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 1 + 1 + \frac{1}{ab} = 2 + 1 + 1 + \frac{1}{2} = \frac{9}{2}
\]

45. A man wants to board a plane with a 5ft long steel rod, but airline regulations say that the maximum length or width (or both) of any object or parcel checked on board is 4 ft. Without bending or cutting the rod, or altering it in any way, how did the man check it through without violating the rule?

Wrap it in a box 3 feet wide and 4 feet long (diagonal is 5ft).

46. From Bhaskara (ca. 1120) “One third of a collection of beautiful water lilies is offered to Mahadev, one-fifth to Huri, one-sixth to the Sun, one-fourth to Devi, and six which remain are presented to the spiritual teacher.” Find the total number of lilies.

\[
x = \frac{1}{3} x + \frac{1}{5} x + \frac{1}{6} x + \frac{1}{4} x + 6
\]

Solution is : 120

47. **Two numbers.** Find the two positive numbers such that their sum equals 4 times their difference and their product equals 9 times their ratio.

\[
x + y = 4(x - y)
\]

\[
x \cdot y = 9 \cdot \frac{x}{y}
\]

, Solution is: \([x = 5, y = 3]\)
48. A town has a population of 20,000 people. \( x \) percent of them are one-legged, and half of the others go barefoot. How many sandals are worn in the town?

\[
x \% \text{ percent are one legged - one sandal} \\
\frac{1}{2} (20000 - x) \text{ go barefoot - zero sandals} \\
\frac{1}{2} (20000 - x) \text{ wear sandals - two sandals} \\
1 \cdot x + 0 \cdot \frac{1}{2} (20000 - x) + 2 \cdot \frac{1}{2} (20000 - x) = 20000
\]

49. Rolly wishes to secure his dog with an 8-foot rope to a square shed that is 16 feet on each side. His preliminary drawings are shown.

Which of these arrangements gives the dog the greater area to roam, and by how many square feet?

(A) I, by \(8\pi\)  
(B) I, by \(6\pi\)  
(C) II, by \(4\pi\)  
(D) II, by \(8\pi\)  
(E) II, by \(10\pi\)

The regions in which the dog can roam for each arrangement are shaded in the figure. For arrangement I, the area of this region is \(\frac{1}{2} \pi \cdot 8^2 = 32\pi \) square feet. The area of the shaded region in arrangement II exceeds this by the area of a quarter-circle of radius 4 feet, that is, \(\frac{1}{4} \pi \cdot 4^2 = 4\pi \) square feet.

50. A paddler in his canoe on a river leaves the dock and paddles upstream for one mile. At that point he encounters a log floating downstream. He continues paddling upstream for one hour. He then turns around and paddles downstream until he returns to the dock just in time to meet up with the log. How fast is the river flowing?

\( v \) is paddler’s speed, \( r \) is river’s speed

\[ T H E N : \] It takes 1 hour for the paddler to travel up the river (before turning)

\[ d \text{ miles } = (v - r) \cdot 1 : d = v - r : \text{ EQUATION (2.1)} \]

The paddler then travelled down the river (with the current) for \(1 + d\) miles

\[ 1 + d = (v + r) \cdot t : 1 + d = vt + rt : \text{ EQUATION (2.2)} \]

During this time, the log traveled at the speed of the current for \(1 + t\) hours

\[ 1 \text{ mile } = r \cdot (1 + t) = r + rt : \text{ EQUATION (2.3)} \]

Add equations (2.1) and (2.3) subtract equation (2.2)

\[ d + 1 - (1 + d) = (v - r + r + rt) - (vt + rt) \] or \(0 = v - vt = v(1 - t) : t = 1\)

So, the paddler travelled up the river for another 1 hour.

So, the log travelled down the river for 2 hours, and it travelled 1 mile, so it’s speed was

\[ r = \frac{1}{2} \text{ mph} \]
51. What is the area enclosed by the graph of \(|3x| + |4y| = 12|?\)

(A) 6 (B) 12 (C) 16 (D) 24 (E) 25

\(|3x| + |4y| = 12\)

\[
\begin{array}{c|c}
\text{X} & \text{Y} \\
-4 & -2 \\
\hline
& \\
2 & 4 \\
\hline
4 & \\
\end{array}
\]

\[A = 4 \left(\frac{1}{2}\right) (4) (3) = 24\]

52. Let \(a, b, c, d,\) and \(e\) be distinct integers such that

\((6 - a)(6 - b)(6 - c)(6 - d)(6 - e) = 45.\)

What is \(a + b + c + d + e = ?\)

(A) 5 (B) 17 (C) 25 (D) 27 (E) 30

\[45 = \pm 1 \cdot \pm 3 \cdot \pm 3 \cdot \pm 5: \text{five distinct numbers, so it must be} \ (-3) \text{AND} (-1) \text{AND} (3) \text{AND} (1) .\]

So, we must have: \((+5)(+3)(-3)(+1)(-1)\) or \(a = 1, b = 3, c = 9, d = 5, e = 7\)

\[1 + 3 + 9 + 5 + 7 = 25\]

53. If \(a\) is a nonzero integer and \(b\) is a positive number such that \(ab^2 = \log_{10} b,\) what is the median of the set \(\{0, 1, a, b, \frac{1}{b}\}\)?

SOLUTION:

Case 1: \(0 < b < 1, \text{so} \ \log_{10} b < 0. \text{Then} \ a = \frac{1}{b^2} \log_{10} b < 0 \text{too, so} \ a < 0 < b < 1 < \frac{1}{b}\)

: Median = \(b\)

Case 2: \(b \geq 1, \text{so} \ \log_{10} b > 0. \text{For} b > 1, \ \log_{10} b < b \text{Then} \ a = \frac{1}{b^2} \log_{10} b > 0 \text{and} \ a = \frac{1}{b} \log_{10} b < 1\)

: Then \(a\) can’t be integer. So - this case is impossible!

54. Suppose that \(\sin a + \sin b = \sqrt{\frac{5}{3}}\) and \(\cos a + \cos b = 1.\) What is \(\cos (a - b)?\)

\[
\begin{align*}
\cos (a - b) &= \cos a \cos b + \sin a \sin b \\
\left(\sqrt{\frac{5}{3}}\right)^2 &= (\sin a + \sin b)^2 = \sin^2 a + 2 \sin a \sin b + \sin^2 b \\
(1)^2 &= (\cos a + \cos b)^2 = \cos^2 a + 2 \cos a \cos b + \cos^2 b \\
\frac{5}{3} + 1 &= 1 + 2 \cos (a - b) + 1 \\
2 \cos (a - b) &= \frac{2}{3}, \text{so} \ \cos (a - b) = \frac{1}{3}
\end{align*}
\]
55. Ann and Barbara were comparing their ages and found that Barbara is as old as Ann was when Barbara was as old as Ann had been when Barbara was half as old as Ann is. If the sum of their present years is 44 years, how old are they now?

SOLUTION: If \(a = Ann\) and \(b = Barbara\), define \(d = a - b\): Ann is \(d\) years older. When Barb was \(\frac{1}{2}a\) years old, Ann was \(\frac{1}{2}a + d\) years old; when Barb was \(\frac{1}{2}a + d\), Ann was \(\frac{1}{2}a + 2d\) old. Barb is now \(\frac{1}{2}a + 2d\) old: \(b = \frac{1}{2}a + 2d\)

\[
\begin{align*}
a + b &= 44 \\
a - b &= d \\
b &= \frac{1}{2}a + 2d
\end{align*}
\]

, Solution is: \([a = 24, b = 20, d = 4]\).

56. Three circles of radius \(s\) are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius \(r > s\) is tangent to both axes and to the second and third circles. What is \(r/s\)?

\[
\begin{align*}
AB &= 3s \\
AC &= r - 3s \\
AD &= r + s \\
CD &= r - s
\end{align*}
\]

\[
\begin{align*}
(r - 3s)^2 + (r - s)^2 &= (r + s)^2 \\
\left(\frac{r}{s} - 3\right)^2 + \left(\frac{r}{s} - 1\right)^2 &= \left(\frac{r}{s} + 1\right)^2 \\
\left(\frac{r}{s}\right)^2 - 6 \left(\frac{r}{s}\right) + 9 + \left(\frac{r}{s}\right)^2 - 2 \left(\frac{r}{s}\right) + 1 &= \left(\frac{r}{s}\right)^2 + 2 \left(\frac{r}{s}\right) + 1 \\
\left(\frac{r}{s}\right)^2 - 10 \left(\frac{r}{s}\right) + 9 &= 0 \\
\left(\frac{r}{s} - 1\right) \left(\frac{r}{s} - 9\right) &= 0 \\
\frac{r}{s} &= 9
\end{align*}
\]
57. A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a lune. Determine the area of this lune.

\[ \text{(A)} \frac{1}{2} \pi - \frac{\sqrt{3}}{4} \quad \text{(B)} \frac{\sqrt{3}}{4} - \frac{1}{12} \pi \quad \text{(C)} \frac{\sqrt{3}}{4} - \frac{1}{24} \pi \quad \text{(D)} \frac{\sqrt{3}}{4} + \frac{1}{24} \pi \quad \text{(E)} \frac{\sqrt{3}}{4} + \frac{1}{12} \pi \]

The dark wedge = sector (radius =1, angle =\( \frac{\pi}{3} \)) - triangle (base =1, height = \( a \))

\[ = \frac{1}{2} (1)^2 \pi \left( \frac{\pi}{3} \right) - \frac{1}{2} (1) \left( \frac{\sqrt{3}}{2} \right) = \frac{\pi}{6} - \frac{\sqrt{3}}{4} \]

Area of the lune = \( \left( \frac{\pi}{8} \right) - \left( \frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) = \frac{3\sqrt{3}}{4} - \frac{1}{24} \pi \)

58. Equal circles are arranged in a regular pattern throughout the plane so that each circle touches six others. What percentage of the plane is covered by circles?

\[ \text{Area of the } \bigodot \text{ in } \bigtriangleup = \frac{3 \times \left( \frac{\pi}{2} \left( \frac{r}{\pi} \right)^2 \right)}{\frac{1}{2} \left( 2r \right) \left( \sqrt{(2r)^2 - r^2} \right)} = \frac{\pi}{2\sqrt{3}} \approx 0.907 = 90.7\% \]
59. The vertices of a 3-4-5 right triangle are the centers of three mutually externally tangent circles as shown. What is the sum of the areas of these circles?

\[ \text{Solution is: } [a = 1, b = 2, c = 3] \]
\[ \text{Area } = a^2 \pi + b^2 \pi + c^2 \pi = (1 + 2^2 + 3^2) \pi = 14 \pi \]

60. A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

\[ \text{Surface of the outside cube } = 6x^2 = 24, \text{ so } x^2 = 4, \text{ } x = 2. \]
\[ \text{radius of sphere } r = \frac{x}{2} = 1, \]
\[ \text{Big diagonal of inside cube } = 1, 1 = \left( \frac{a}{2} \right)^2 + \left( \frac{a}{2} \right)^2 + \left( \frac{a}{2} \right)^2 = \frac{3}{4}a^2 : a^2 = \frac{4}{3} \]
\[ \text{Surface of the inside cube } = 6a^2 = 6 \left( \frac{4}{3} \right) = 8 \]

61. Alma, Bess, Cleo, and Dina visited Elf on Saturday. Alma visited at 8:00, Bess visited at 9:00, Cleo visited at 10:00, and Dina visited at 11:00. Some clues:
   i) The times may be either A.M. or P.M.
   ii) At least one woman visited Elf between Alma and Bess.
   iii) Cleo or Dina (or both) visited Elf before Alma did.
   iv) Cleo did not visit Elf between Bess and Dina.
Who visited Elf last?

Solution:
Since Cleo or Dina visited Elf BEFORE Alma - Alma had to have visited at 8:00 PM!
Since At least one woman was between Alma and Bess, Bess had to have visited at 9:00 AM!
Since Cleo Did NOT visit between Bess and Dina: If Dina was PM -> EVERYBODY is between Bess and Dina, so Dina MUST have been AM.
Since Cleo Did NOT visit between Bess and Dina: Cleo must have been PM. and CLEO is LAST!
62. A runner runs a course at constant speed of 6mph. One hour after the runner begins, the cyclist starts in the same course at a constant speed of 15mph. How long after the runner starts does the cyclist overtake the runner?

Runner at 6mph for $t$ hours distance = $6t$

One hour later cyclist at 15 mph for $(t - 1)$ hours distance = $15(t - 1)$

$6t = 15(t - 1)$, Solution is: $\frac{5}{3}$ hours = 1 hour and 40 minutes

63. Three people check into a hotel. They pay $30 to the manager and go to their room. The manager finds out that the room rate is $25 and gives $5 to the bellboy to return. On the way to the room the bellboy reasons that $5 would be difficult to share among three people so he pockets $2 and gives $1 to each person.

Now each person paid $10 and got back $1. So they paid $9 each, totalling $27. The bellboy has $2, totalling $29.

Where is the remaining dollar? Explain your answer in detail!

There are two ways to calculate total: by posession and by intent.

<table>
<thead>
<tr>
<th>Manager keeps $25</th>
<th>Bellboy kept $2</th>
<th>Each roommate has $1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>TOTAL: $30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>They paid $9 each</th>
<th>They kept $1 each</th>
<th>TOTAL: $30</th>
</tr>
</thead>
</table>

64. If a parade 2 miles long is proceeding at 3 m.p.h., how long will it take a runner, jogging at 6 m.p.h., to travel from the front of the parade to the end of the parade?

parade is moving at 3 mph for $t$ hours $dp=$distance parade moves = $3t$

runner is moving at 6mph in the opposite direction $dr=$distance runner moves = $6t$

runner has to move 2 miles more $6t = 3t + 2$

Solution: $t = \frac{2}{3}$ hour = 40 minutes

65. What is the length of the line PQ drawn in the unit circle?

What is the length of the line PQ drawn in the unit circle?

**SOLUTION**

PQ is one of the diagonals of a rectangle, and is therefore the same as the other diagonal. The other diagonal is the radius of this unit circle, and is therefore 1. Hence, length of the line PQ is exactly 1.
66. Volumes I, II and III of an extraordinary book, *Autobiography of a Genius*, by Iztok Hozo, are standing in numerical order from left to right on your bookshelf. Volume I has 1875 pages, Volume II has 55 pages (what happened in grad school stays in grad school) and Volume III has 5897 pages (yes it is huge). A bookworm eats from the first page (Vol I, page 1) of the autobiography to the last page (Vol III, page 5897). The bookworm eats in a straight line. Not counting covers, title pages, etc., how many pages does the bookworm eat through?

Make a picture: only volume II is eaten through.

67. If you had a piece of paper that was 0.001 inch thick, how tall a pile would it make if it was doubled (folded) fifty times? Compare your number with the distance from earth to the moon. Which is larger?

If you double it once: the thickness is $0.001 \times 2$
If you double it twice: the thickness is $0.001 \times 2^2$
If you double it ten times: the thickness is $0.001 \times 2^{10}$
If you double it fifty times: the thickness is $0.001 \times 2^{50}$

1 mile = 5280 feet = 5280 · 12 inches = 63,360 inches
$0.001 \times 2^{50}$ inches $= \frac{0.001 \times 2^{50}}{63360}$ miles $\approx 17,769,885$ miles

Note: Moon is approximately 250,000 miles away

68. When you travel to work going 60 mph, you arrive there early. When you travel to work going 30 mph, you arrive there late. The amount of time you are early is the same as the amount of time you are late. How fast should you go to get to work exactly on time?

**Solution:**

<table>
<thead>
<tr>
<th></th>
<th>Speed</th>
<th>Time</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Early</td>
<td>60 mph</td>
<td>$\frac{x}{60}$ hours</td>
<td>$x$ miles</td>
</tr>
<tr>
<td>Late</td>
<td>30 mph</td>
<td>$\frac{x}{30}$ hours</td>
<td>$x$ miles</td>
</tr>
</tbody>
</table>

Since the average of the two times is exactly right, the perfect time is $t = \frac{\frac{x}{60} + \frac{x}{30}}{2} = \frac{x}{40}$; therefore, you should drive at 40 mph to arrive exactly on time.

69. **Ask Marilyn**

By Marilyn vos Savant Published: January 16, 2005

At 60 mph, it takes 60 seconds to travel a mile. At 120 mph, it takes 30 seconds. At what speed would it take 45 seconds?

$$x \text{ mph} = \frac{x \text{ miles}}{1 \text{ hour}} = \frac{x \text{ miles}}{3600 \text{ seconds}} = \frac{1 \text{ mile}}{\frac{3600}{x} \text{ seconds}}$$

$$\frac{3600}{x} = 45, \text{ Solution is : 80 mph}$$
70. A driver sets out on a 20-mile trip.
   (a) When he has gone halfway he finds he has averaged 25 mph. At what speed must he travel the rest of the way to make his overall average speed for the trip 40 mph?

\[
\text{Halfway} = 10 \text{ miles at } 25 \text{ mph} : 10 = \frac{10}{25} = \frac{2}{5} = 0.40 \text{ hours} \\
\text{Complete trip} = 20 \text{ miles at } 40 \text{ mph} : 20 = \frac{20}{40} = \frac{1}{2} = 0.50 \text{ hours} \\
\text{So, the second part of the trip} = 10 \text{ miles in } 0.10 \text{ hours} : 10 = \frac{10}{0.10} = 100 \text{ mph}
\]
   The speed for the second part of the trip must be 100 mph

(b) When he has gone halfway he finds he has averaged 20 mph. At what speed must he travel the rest of the way to make his overall average speed for the trip 40 mph?

\[
\text{Halfway} = 10 \text{ miles at } 20 \text{ mph} : 10 = \frac{10}{20} = \frac{1}{2} = 0.50 \text{ hours} \\
\text{Complete trip} = 20 \text{ miles at } 40 \text{ mph} : 20 = \frac{20}{40} = \frac{1}{2} = 0.50 \text{ hours} \\
\text{So, the second part of the trip} = 10 \text{ miles in } 0 \text{ hours} \quad \text{IMPOSSIBLE!!!}
\]

71. Say that two motorboats are on opposite shores of a river moving toward each other, but at different speeds. (Neglect all other factors, like acceleration, turn-around and current.) When they pass each other the first time, they are 700 yards from one shoreline. They continue to the opposite shore, then turn around and start moving toward each other again. When they pass the second time, they are 300 yards from the other shoreline. (Their speeds, although different, remain constant.) How wide is the river?

\[
\text{Time is } t_1 \text{ when they meet the first time.} \\
\text{Second boat travelled 700 yards: } v_2 t_1 = 700 : \frac{700}{v_2} \\
\text{The first boat travelled } 300 + d \text{ yards: } v_1 t_1 = 300 + d : t_1 = \frac{300 + d}{v_1} \\
\text{Proportion: } \frac{700}{v_2} = \frac{300 + d}{v_1}, \text{ or } \frac{v_1}{v_2} = \frac{300 + d}{700} \\
\text{The second time they meet, is after the time } t_2 \\
\text{The first boat travelled } 700 + 700 + d \text{ yards: } 1400 + d = v_1 t_2 : t_2 = \frac{1400 + d}{v_1} \\
\text{The second boat travelled } d + 300 + 300 \text{ yards: } d + 600 = v_2 t_2 : t_2 = \frac{d + 600}{v_2} \\
\text{Proportion: } \frac{1400 + d}{v_1} = \frac{d + 600}{v_2} \text{ or } \frac{v_1}{v_2} = \frac{1400 + d}{600 + d} \\
\text{Make them equal: } \frac{300 + d}{700} = \frac{1400 + d}{600 + d}. \text{ Solution is: } -1000, 800 \\
\text{Therefore } d = 800 \text{ yards, and the river is } 300 + 800 + 700 = 1800 \text{ yards wide!}
\]

72. A car travels downhill at 72 m.p.h. (miles per hour), on the level at 63 m.p.h., and uphill at only 56 m.p.h. The car takes 4 hours to travel from town A to town B. The return trip takes 4 hours and 40 minutes. Find the distance between the two towns.

SOLUTION

\[
\frac{d}{72} + \frac{l}{63} + \frac{u}{56} = 4 \\
\frac{d}{56} + \frac{l}{63} + \frac{u}{72} = 4 + \frac{2}{3}
\]
Subtract first from the second (note that $\frac{1}{56} - \frac{1}{72} = \frac{1}{252}$):

$$-\frac{1}{252}d + \frac{1}{252}u = \frac{2}{3}$$
$$u = 168 + d$$

Plug it in the first equation:

$$\frac{d}{72} + \frac{l}{63} + \frac{168 + d}{56} = 4$$

Solve it for $l$:

$$l = 63 - 2d$$

The distance between the towns is: $d + l + u = d + (63 - 2d) + (168 + d) = 231$ miles.

73. One of my students told me she will be exactly $x$ years old in the year $x^2$. How old is this student this year (2013)?

$$45^2 = 2025 : \text{Born 1980, This year 33 years old!}$$

74. The following four statements, and only these, are found on a card:

- On this card exactly one statement is false.
- On this card exactly two statements are false.
- On this card exactly three statements are false.
- On this card exactly four statements are false.

Assume each statement is either true or false. Is any of these statements a true statement? Explain your answer in detail!

**Answer:** There are five statements - they are contradicting, so at most one can be true! Since one MUST be true that means four must be false! Hence, "On this card exactly four statements are false." is the true statement.

75. A woman, her brother, her son and her daughter are chess players (all relations by birth). The worst player’s twin (who is one of the four players) and the best player are of opposite sex. The worst player and the best player are the same age. Who is the worst player?

**Answer:** The worst player and the best player are the same sex, and the same age (two choices: male or female). Mother and daughter can’t be the same age - so not female. The brother and the son can be the same age. So one is best player and the other is the worst player. But in that case - brother can’t be twin to the mother - so the worst player must be the son, and the best player is the brother!

76. A man can commute either by train or by bus. He always goes to work on the train in the morning, he comes home on the train or on the bus in the afternoon. During a total of $x$ working days, the man commuted by the bus 8 times, and commuted by train 26 times. Find $x$.

**Answer:** He took ONLY train (twice per day): $26 - 8 = 18$ times $\implies 18/2 = 9$ days; the other 8 days he took train + bus: Total = 17 days.

OR
He travels twice per day; total number of travels (bus or train) = $26 + 8 = 34$; so number of days is $34/2 = 17$.

77. If the whatsis is so when whosis is is and the so and so is is · so, what is the whosis · whatsis when the whosis is so, the so and so is so · so, and the is is two (whatis, whosis, is and so are positive)?

(a) whosis · is · so  (b) whosis  (c) is
(d) so  (e) so and so
Answer: what is \( a \); \( \text{whosis} = b; \text{is} = c; \text{so} = d \), then

\[
\begin{align*}
(b = c \text{ and } d + d = c \cdot d) & \implies (a = d) \\
(b = d \text{ and } d + d = d \cdot d \text{ and } c = 2) & \implies (a \cdot b =?)
\end{align*}
\]

Rephrase the first equation: \( 2d = c \cdot d \) is the same as \( c = 2 \), and then so is \( b = 2 \). Therefore: if \( b = c = 2 \) then \( a = d \). In the second list of assumptions, if \( c = 2 \) and \( 2d = d^2 \), so \( d = 2 \), and \( b = d = 2 \), then \( a \cdot b = 2a = 2d = d + d = so \) and so. Answer (e).

78. Many years ago, a cruise liner sank in the middle of the Pacific ocean. The survivors luckily landed on a remote desert island. There was enough food for the 220 people to last three weeks. Six days later a rescue ship appeared, unluckily this ship also sank, leaving another 55 people stranded on the island to now share the original rationed food. The food obviously had to be re-rationed, but everyone was now on one-half of the original ration, so how many days in total would the food last for, from the day of the original sinking?

Let \( R \) = Quantity of the original food : Original ration = \( \frac{R}{220 \text{ people} \cdot 21 \text{ days}} \)

After six days, the amount of food is : \( R - \left(6 \text{ days} \cdot 220 \text{ people} \cdot \frac{R}{220 \cdot 21}\right) = \frac{5}{7}R \)

New ration is \( \frac{1}{2} \) of the original ration = \( \frac{1}{2} \frac{R}{220 \cdot 21} \)

Number of extra days is \( t \) : \( \frac{1}{2} \frac{R}{220 \cdot 21} \cdot t \text{ days} \cdot ((220 + 55) \text{ people}) = \frac{5}{7}R \) : Solution is \( t = 24 \)

Total number of days : \( t = 24 + 6 = 30 \text{ days} \)

79. On a railway track we find a tunnel, which is 5 miles long. A train, 440 yards long, enters the tunnel at a speed of 50 miles per hour. How long will it take for the whole of the train to pass completely through the tunnel?

1 mile = 1760 yards : The train needs to travel \( 5 + \frac{440}{1760} = 5.25 \) miles

Time needed to travel is : \( \frac{5.25}{50} = 0.105 \) hours = 6.3 minutes = 6 min and 18 sec = 378 seconds

80. What does this equation simplify to?

\[
(x - a) (x - b) (x - c) (x - d) \cdots (x - z) =
\]

\[
(x - a) (x - b) (x - c) (x - d) \cdots (x - x) (x - y) (x - z) = 0
\]

This one is zero!

81. A man is jogging accross a bridge. When he is 3/8ths of the way across, he hears a train coming from behind him. He calculates that if he keeps running, he will reach the end of the bridge at the same instant as the train. He also calculates that if he turns back and runs back, he will reach the beginning of the bridge at the same instant as the train. Say the man runs constantly at 8 mph. What is the speed of the train?

SOLUTION

The bridge is \( d \) miles long. So, if the man runs back towards the beginning of the bridge, he has 3/8ths to go at 8 mph, the time is

\[
t = \frac{3}{8}d = \frac{3}{64} \text{ hours}
\]

If he runs toward the end of the bridge, the time he needs will be

\[
t = \frac{5}{8}d = \frac{5}{64} \text{ hours}
\]

The train is moving at \( x \) miles per hour. Say train is \( a \) miles from the beginning of the bridge. Since the train would arrive at the same time as the man, to both, the beginning and to the end of the bridge, we have:

\[
x \left( \frac{3}{64}d \right) = a
\]

\[
x \left( \frac{5}{64}d \right) = a + d
\]
Subtract the top equation from the bottom one:

\[ x \left( \frac{2}{64}d \right) = d \]

Divide by \( d \), and solve for \( x \):

\[ x = 32 \text{ mph} \]

The train is going at 32 miles per hour.

You can also solve this problem using just a logic argument:
Suppose the man ran towards the end of the bridge. At he passes another 3/8ths of the length of the bridge towards the end (so he has only 2/8ths to go), the train arrives at the beginning of the bridge. So he has 1/4th of the length to run and arrives at the end at the exact same moment as the train. Therefore the train is 4 times faster: \( 4 \times 8 = 32 \text{ mph} \).

82. Show that

\[ \cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{3\pi}{5} \cos \frac{4\pi}{5} = \frac{1}{16} \]

**SOLUTION:**

Multiply both sides with \( \sin \frac{\pi}{5} \):

\[ \sin \frac{\pi}{5} \cos \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \frac{3\pi}{5} \cos \frac{4\pi}{5} = \frac{1}{16} \sin \frac{\pi}{5} \]

Now use the trigonometric identity \( \sin \alpha \cos \alpha = \frac{1}{2} \sin 2\alpha \) repeatedly (three times) for \( \alpha = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{4\pi}{5} \) and obtain

\[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \cos \frac{3\pi}{5} \sin \frac{8\pi}{5} = \frac{1}{16} \sin \frac{\pi}{5} \]

Use the trigonometric identity \( \cos \alpha \sin \beta = \frac{1}{2} (\sin (\alpha + \beta) - \sin (\alpha - \beta)) \), so \( \sin \frac{8\pi}{5} \cos \frac{3\pi}{5} = \frac{1}{2} \left( \sin \frac{11\pi}{5} - \sin \frac{\pi}{5} \right) \).

Note that \( \left( \frac{1}{2} \right)^4 = \frac{1}{16} \), \( \sin (-\pi) = 0 \), and \( \frac{11\pi}{5} = 2\pi + \frac{\pi}{5} \) the equation becomes:

\[ \sin \left( 2\pi + \frac{\pi}{5} \right) = \sin \frac{\pi}{5} \]

which is, of course, true: the identity holds!
Pigeonhole principle
The problems below use the following simple, but very powerful principle:

**Pigeonhole Principle:**
"If a flock of $n$ pigeons comes to roost in a house with $r$ pigeonholes and $n > r$, then at least one hole contains more than one pigeon."

83. Prove that if 5 pins are stuck onto a piece of cardboard in the shape of an equilateral triangle of side length 2, then some pair of pins must be within distance 1 of each other.

**SOLUTION:**

![Equilateral Triangle Diagram]

Divide the equilateral triangle into 4 equilateral triangles with sides of length 1 each. Then, one of these four triangles must contain two of the five points. Those two points are at most 1 apart from each other. DONE!

84. If 9 people are seated in a row of 12 chairs, then some consecutive set of 3 chairs are filled with people.

**SOLUTION:**

![Chairs Diagram]

Turn the question around: There are 9 people and 12 chairs. So 3 chairs will remain empty. Divide all the chairs in four groups of 3 (like the figure above). Since we have three empty chairs and four groups of chairs, at least one group of chairs will have no empty chairs. DONE!

85. Show that from any five integers, not necessarily distinct, one can always choose three of these integers, whose sum is divisible by 3.

**SOLUTION:**

Divide the integers into three groups $G_0 = \text{divisible by 3}$, $G_1 = \text{remainder is 1 when dividing by 3}$, $G_2 = \text{remainder is 2 when dividing by 3}$.

By Pigeonhole principle, either (1) each of these groups has at least one integer → the sum of these three is divisible by 3, OR (2) one group is empty so at least one group has THREE numbers in it - those three have sum that is divisible by 3.

86. Let $Z^2 = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{Z}\}$ be the lattice of points with integer coordinates in the real numbers plane. If we select five points in $Z^2$, why is it certain that at least one mid-point of a segment joining a pair of chosen points also belongs to $Z^2$?

**SOLUTION:**

The midpoint of a segment $(x_1, y_1)(x_2, y_2)$ has coordinates $(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Therefore, the midpoint will be in $Z^2$ iff $(x_1 + x_2)$ and $(y_1 + y_2)$ are both even. That will happen when both $x$’s are even or both odd and both $y$’s are even or both odd.

There are four possible even-odd combinations for a point $(x, y)$

$I : (\text{even,even}); \ II : (\text{even,odd}); \ III : (\text{odd,even}); \ IV : (\text{odd,odd})$

Since we have five points, at least two will be in the same group. Those two points will have the midpoint of their segment in $Z^2$. DONE!
87. A railroad bridge is 20 ft above, and at right angles to, a river. A man in a train traveling 60 miles per hour passes over the center of the bridge at the same instant a man in a motorboat traveling 20 miles per hour passes under the center of the bridge. Has fast are the two men moving away from each other 1 second later?

Boat and the bridge make a right triangle: $D_{\text{boat}}^2 + 20^2 = c^2$
The Train track and the boat make a right triangle: $D_{\text{man}}^2 + c^2 = \text{Distance}^2$
Combine the two: $D_{\text{boat}}^2 + 20^2 + D_{\text{man}}^2 = \text{Distance}^2$

Implicit Differentiation:

or \[
\frac{d (\text{Distance})}{dt} = \frac{2D_{\text{boat}} \frac{dD_{\text{boat}}}{dt} + 0 + 2D_{\text{man}} \frac{dD_{\text{man}}}{dt}}{\text{Distance}} = \frac{D_{\text{boat}} v_{\text{boat}} + D_{\text{man}} v_{\text{man}}}{\text{Distance}}
\]

\[\begin{align*}
v_{\text{man}} &= \frac{dD_{\text{man}}}{dt} = 60 \text{ mph} = \frac{60}{\frac{5280}{3600}} = 88 \text{ ft/sec}. \text{ Therefore } D_{\text{man}} &= 88 \text{ feet}. \\
v_{\text{boat}} &= \frac{dD_{\text{boat}}}{dt} = 20 \text{ mph} = \frac{20}{\frac{5280}{3600}} = \frac{88}{3} \text{ ft/sec}. \text{ Therefore } D_{\text{boat}} &= \frac{88}{3} \text{ feet}
\end{align*}\]

Distance \[= \sqrt{D_{\text{boat}}^2 + 20^2 + D_{\text{man}}^2} = \sqrt{\left(\frac{88}{3}\right)^2 + 20^2 + (88)^2} = 94.892\]
Plug it all in: \[
\frac{d (\text{Distance})}{dt} = \frac{(88)(88) + \left(\frac{88}{3}\right)^2}{3600} = \frac{94.892}{5280} = 90.676 \text{ ft/sec}
\]

88. A group of functions are walking down the street, calmly talking to each other, strolling to a local bar. Suddenly the Derivative pops up from behind a corner and screams at them: "Now you are in biggg trouble. I'll come over there and differentiate each one of you!" "Oh, no", screams Tanx ( \(\tan x\)), "I really don't want to turn into Square Secan." "Help, help", yells Ex Cube. "I don't want to lose my power status. I'll be reduced to a Quadratic."
For some reason, however, one of the functions calmly kept walking down the street. She didn't even inch when the derivative came over. She was not afraid.
WHY? What was her name?
SOLUTION: \(y = e^x\) : name is Etta Thex (or something with similar sound)

89. You drop a pebble off a bridge into a stream. Beginning with an initial velocity of 0, it falls faster and faster under the pull of gravity, but when it hits water it slows down to a constant terminal velocity, which it maintains until it hits the bottom. Sketch a graph of the pebble’s distance from your hand as a function of time to the time
90. A basketball player can jump 31 inches vertically. How much time does the player spend in the top 6 inches of the jump and how much time in the bottom 6 inches of the jump? In your own words, explain why basketball players seem to be suspended in air when they jump. (Note: expect messy numbers - round to two or three decimal places if you want to).

SOLUTION:
The equation describing the height of a jump is \( h(t) = v_0t - \frac{1}{2}gt^2 \), where \( v_0 \) is the initial velocity of the jump and \( g = 32 \text{ ft/s}^2 \) is the gravitational constant. Therefore, \( h'(t) = v_0 - gt \), and the maximum will occur when \( h'(t) = 0 \), or \( t = \frac{v_0}{g} \). The highest point is then \( h\left(\frac{v_0}{g}\right) = v_0 \left(\frac{v_0}{g}\right) - \frac{1}{2}g \left(\frac{v_0}{g}\right)^2 = \frac{1}{2}v_0^2 \). Since we know the highest point was 31 inches = \( \frac{31}{12} \) feet, we can solve the following equation for \( v_0 \):

\[
\frac{1}{2}v_0^2 = \frac{31}{12},
\]

Solution is:

\[
v_0 = \frac{4}{3}\sqrt{93} \text{ ft/sec} = 12.858 \text{ ft/sec}.
\]

Now, we have the equation for the height:

\[
h(t) = \frac{4}{3}\sqrt{93}t - 16t^2.
\]

The top 6 inches of the height will occur between the two solutions of the equation \( h(t) = \frac{25}{12} \),

\[
\frac{4}{3}\sqrt{93}t - 16t^2 = \frac{25}{12},
\]

Solutions are: \( t = \frac{1}{24}\sqrt{93} - \frac{1}{8}\sqrt{2} = 0.22504 \) seconds, and \( t = \frac{1}{8}\sqrt{2} + \frac{1}{24}\sqrt{93} = 0.57860 \) seconds. So the time spent in the top 6 inches of the jump is \( t = 0.5786 - 0.225 = 0.3536 \) seconds.

On the other hand, the time spent in the bottom 6 inches of the jump is the time from 0 to the first time when \( h(t) = \frac{6}{12} \). (twice - to account for the trip down as well).

\[
\frac{4}{3}\sqrt{93}t - 16t^2 = \frac{6}{12},
\]
The first solution is: \( t = \frac{1}{24} \sqrt{93} - \frac{5}{24} \sqrt{3} = 0.04 \) seconds, so the basketball jumper spends \( 2(0.04) = 0.08 \) seconds in the bottom part of the jump, compared to 0.35 (five times longer) in the top 6 inches.

91. Calumet river in the Sunnyvale city has parallel banks with the distance between them 100 yards. It has some islands with the total perimeter 800 yards. Mr. Know–It-All claims that it is possible to cross the river in a boat from an arbitrary point in such a way so that the total trajectory across the river will not exceed 300 yards. Is he right? EXPLAIN! What is hypothetically the longest possible path across this river (to the nearest yard)?

**SOLUTION:**

In the worst case scenario, the islands are of negligible width and the longest length can be 400 yards. If the starting point is anywhere but exactly at the middle of the island, we will simply pick the shorter way, so the longest possible path will start at the very middle (at most 200 yards). Then, the total distance

\[
D(x) = \sqrt{200^2 + x^2} + (100 - x)
\]

Let’s maximize the function \( D(x) \), for \( x \in [0, 100] \)

\[
D'(x) = \frac{2x}{2\sqrt{200^2 + x^2}} - 1 = \frac{x - \sqrt{200^2 + x^2}}{\sqrt{200^2 + x^2}}
\]

So \( D'(x) = 0 \), when \( x - \sqrt{200^2 + x^2} = 0 \), or \( x = \sqrt{200^2 + x^2} \), \( x^2 = 200^2 + x^2 \) - which has no solutions. Therefore, the extrema of the function \( D(x) \) are at the endpoints: \( D(0) = \sqrt{200^2 + 100} = 300 \) or \( D(100) = \sqrt{200^2 + 100^2 + 0} = 100\sqrt{5} \approx 223.66 \). Therefore, the longest trip is 300 yards!

**Probability**

92. *Teachers’ diligence finds fame, free lunch.* The Morning Call (Allentown), Jan. 21, 1995, B1 Joseph P. Ferry

The article states that Bob Swain, a Souderton High School mathematics teacher, found a mistake in an ad of the restaurant chain Boston Chicken. With much effort, he got them to change the ad. He received a free lunch with thirty of his students. He also obtained instant fame, appearing, for example, on the CBS program Good Morning America.

Here is what you learn from the article about the mathematics of the problem.

– Boston Chicken’s ad, featuring quarterback Joe Montana, claimed that there are 3,360 possible combinations of the three-item side dishes to accompany your main dish. There are 16 side dishes available.

Swain claimed that whoever had written the ad had confused the formula for figuring permutations and combinations. By Swain’s calculations, there are only 816 different combinations available.

Boston Chicken agreed to change their ad to say 816 possible combinations.

(a) How did Boston Chicken arrive at the number 3360?

\[
P(16, 3) = 16 \cdot 15 \cdot 14 = 3360
\]

(b) How did Swain arrive at the number 816?

\[
C(16, 3) + C(16, 2)C(2, 1) + C(16, 1)
\]

Pick three different side items
Pick two items, one to repeat
Pick one item, repeat it three times

\[
= 560 + 120 \cdot 2 + 16 = 816
\]

A reader poses the following question: "I'm flying over the China Sea in a single-engine plane. The same route is being own by my buddy in a twin-engine plane. The engines are made by the different companies, but they're the same in all other respects, such as age, condition and inherent reliability. It is known that the twin-engine plane cannot maintain flight on a single engine. Our destination is hours away. Which plane has a higher probability of going down because of engine failure?"

Marilyn says the single engine plane is safer, claiming that if all other factors are equal, the twin-engine plane is twice as likely to go down. What is Marilyn forgetting?

\[
P(SE) = P(Engine failure)
\]

\[
P(TE) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)
\]

She is forgetting \(-P(E_1 \cap E_2)\)

94. Galileo and the three dice

Three dice are thrown. What is the chance of getting a total of 9 spots? What about 10 spots?

In the seventeenth century, Italian gamblers (just like the French) used to bet on the total number of spots rolled with three dice. They believed that the chance of rolling a total of 9 ought to equal the chance of rolling the total of 10. For instance, they said, one combination with a total of 9 spots is

1 spot on one die, 2 spots on another die, 6 spots on third die

This can be abbreviated as “1 2 6.” There are altogether six combinations for 9:

126 135 144 234 225 333

Similarly, they found six combinations for 10:

145 136 226 235 244 334

Thus, argued the gamblers, 9 and 10 should by rights have the same chance. However, experience showed that 10 came up a bit more often than 9. They asked Galileo (Italy, 1564—1642) for help with this contradiction, and he solved the problem. Can you explain it?

Total: \(6^3 = 216\) combinations

126 can occur in \(3! = 6\) different ways: 126, 162, 216, 261, 612, 621

225 can occur in \(\frac{3!}{2!} = 3\) different ways: 225, 252, 522

333 can occur in \(\frac{3!}{3!} = 1\) different way: 333

9: \(3! + 3! + \frac{3!}{2!} + 3! + \frac{3!}{2!} + \frac{3!}{3!} = 25\) Combinations

10: \(3! + 3! + \frac{3!}{2!} + 3! + \frac{3!}{2!} + \frac{3!}{3!} = 27\) Combinations

95. “Bertrand’s Box” Problem

A box has three drawers; one contains two gold coins, one contains two silver coins, and one contains one gold coin and one silver coin. Assume that one drawer is selected randomly and that a randomly selected coin from that drawer turns out to be gold. What is the probability that the chosen drawer is one that contains two gold coins?

\[
P(1) = P(2) = P(3) = \frac{1}{3}
\]

\[
P(G|1) = 1 : P(G|2) = \frac{1}{2} : P(G|3) = 0
\]

\[
P(1|G) = \frac{P(G|1)P(1)}{P(G|1)P(1) + P(G|2)P(2) + P(G|3)P(3)}
\]

\[
= \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} = \frac{2}{3}
\]
96. **Ask Marilyn** Parade Magazine (January 26, 1997)
Dear Marilyn: I play the lottery by buying one $1 ticket every day, using the same six numbers each time. The range of numbers is from 1 through 49, inclusive. You win if you match all six numbers drawn that day (in any order). Suppose that I will live long enough to be sure of winning, no matter how long that is. How long will I have to live (in years) to be mathematically certain that my number will come up a winner at least once? —Jack Hatcher, Sarasota, Fla.

\[
\binom{49}{6} = 13,983,816
\]

Expected to win if lives: \(\frac{13,983,816}{365}\) years = 38312 years

Guaranteed to win : \(\infty\) years

97. **Ask Marilyn**
Marilyn vos Savant

Question: Let’s say my friend puts six playing cards face-down on a table. He tells me that exactly two of them are aces. Then I get to pick up two of the cards. Which of the following choices is more likely?
(A) That I’ll get one or both of the aces
(B) That I’ll get no aces

\[
\begin{align*}
C(6, 2) &= 15 \text{ possibilities} \\
P(2 \text{ aces}) &= \frac{1}{15} \\
P(1 \text{ ace}) &= \frac{C(2, 1)C(4, 1)}{C(6, 2)} = \frac{2 \cdot 4}{15} = \frac{8}{15} \\
P(0 \text{ aces}) &= \frac{C(4, 2)}{C(6, 2)} = \frac{6}{15}
\end{align*}
\]

98. **Ask Marilyn**
By Marilyn vos Savant Published: May 1, 2005
A friend is pregnant with twins that she knows are fraternal. What are the chances that at least one of the babies is a girl?—Nancy Swigert, Lexington, Ky.

Marilyn’s Answer: This may be hard to believe, but they’re 75%. Even tougher to accept, the chances that at least one of the babies is a boy are also 75%!

Explain this answer. Do you agree?

\[
Yes: P(\geq 1G) = \frac{\{BG, GB, GG\}}{\{BB, BG, GB, GG\}} = \frac{3}{4} \\
No: P(\geq 1B) = \frac{\{BG, GB, BB\}}{\{BB, BG, GB, GG\}} = \frac{3}{4}
\]

99. John tosses 6 fair coins, and Mary tosses 5 fair coins. What is the probability that John gets more “heads” than Mary?

\[
P(J > M) = \sum_{j=1}^{6} \sum_{m=0}^{j-1} P(J = j \text{ AND } M = m) = \sum_{j=1}^{6} \sum_{m=0}^{j-1} \binom{6}{j} \binom{5}{m} \frac{1}{2^6} \frac{1}{2^5} = \frac{1}{2}
\]

Or, a logical solution: If John has HHHTTT and Mary has HHTTT, then that is a favorable outcome because John has more Heads than Mary. If you flip over all the coins, John will have TTHHHH and Mary will have TTHHH and that is an unfavorable outcome because John will have the same number of Heads as Mary. In general, if they have the same number of Tails, then John must have more Heads, so that is a favorable outcome, but flipping over all the coins will give them the same number of Heads because those were the coins that were Tails before they were flipped over. So, every favorable outcome corresponds to an unfavorable outcome. Hence, the probability is 50% = \(\frac{1}{2}\).