Chapter 6 - Amortization Schedules and Sinking Funds

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1 Finding the Outstanding Loan Balance

The *outstanding loan balance* goes by many names

- Outstanding principal
- Unpaid balance
- Remaining loan indebtedness

Let's analanyze setp-by-step what happens. Suppose we have a loan L with payments P at the end of each year for n years at an annual effective rate of interest i. Then our equation of value is

$$L_0 = Pa_n \tag{1}$$
$$\tag{2}$$

The value of the loan at time $t = 1$ is $L_0(1+i)$. After the end of year, we have the value of the loan L_1 equal to

$$\begin{aligned}
L_1 &= L_0(1+i) - P && (3)\\
&= L_0 - (P - L_0 \cdot i) && (4)\\
&= L_0 - (P - Pa_n \cdot i) && (5)\\
&= L_0 - \left(P - P \cdot \frac{1-\nu^n}{i} \cdot i\right) && (6)\\
&= L_0 - (P - P + P\nu^n) && (7)\\
&= L_0 - P\nu^n && (8)\\
&= \left(P + P\nu + P\nu^2 + \cdots + P\nu^n\right) - P\nu^n && (9)\\
&= P + P\nu + P\nu^2 + \cdots + P\nu^{n-1} && (10)\\
&= Pa_{n-1} && (11)
\end{aligned}$$

At time $t = 2$

$$\begin{aligned}
L_2 &= L_1(1+1) - P && (12)\\
&= (L_0(1+i) - P)(1+i) - P && (13)\\
&= L_0(1+i)^2 - (P(1+i) + P) && (14)\\
&= L_0(1+i)^2 - Ps_2 && (15)
\end{aligned}$$

Also,

$$\begin{aligned}
L_2 &= L_1(1+i) - P && (16)\\
&= L_1 - (P - L_1 \cdot i) && (17)\\
&= L_1 - (P - Pa_{n-1} \cdot i) && (18)\\
&= L_1 - \left(P - P \cdot \frac{1-\nu^{n-1}}{i} \cdot i\right) && (19)\\
&= L_1 - \left(P - P + P\nu^{n-1}\right) && (20)\\
&= L_1 - P\nu^{n-1} && (21)\\
& && (22)
\end{aligned}$$

And finally,

$$\begin{aligned}
L_2 &= L_1 - P\nu^{n-1} && (23)\\
&= \left(P + P\nu + P\nu^2 + \cdots + P\nu^{n-1}\right) - P\nu^{n-1} && (24)\\
&= P + P\nu + P\nu^2 + \cdots + P\nu^{n-2} && (25)\\
&= Pa_{n-2} && (26)
\end{aligned}$$

In general, to determine the loan balance at time $t = k$

$$\begin{aligned}
L &= Pa_n && (27)\\
L\,(1+i)^k &= Pa_n\,(1+i)^k && (28)\\
&= P \cdot \frac{1-\nu^n}{i} \cdot (1+i)^k && (29)\\
&= P \cdot \frac{(1+i)^k - \nu^{n-k}}{i} && (30)\\
&= P \cdot \frac{(1+i)^k - 1 + 1 - \nu^{n-k}}{i} && (31)\\
&= P\left(\frac{(1+i)^k - 1}{i} + \frac{1-\nu^{n-k}}{i}\right) && (32)\\
&= Ps_k + Pa_{n-k} && (33)\\
L(1+i)^k - Ps_k &= Pa_{n-k} && (34)
\end{aligned}$$

Therefore, we can use any of the following three methods to determine the otstanding balance of the loan at time $t = k$

1. ***Prospective Method*** $L_k = Pa_{n-k}$

2. ***RetrospectiveMethod*** $L_k = L_0(1+i)^k - Ps_k$

3. ***The previous balance*** $L_k = L_{k-1} - P\nu^{n-k+1}$

2 Amortization Schedule

An ***amortization schedule*** is a table which shows the division of each payment between principal and interest, as well as the otstanding loan balance.

2.1 Construction of the Amortization Schedule

Period	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				a_n
1	1	$ia_n = 1 - \nu^n$	ν^n	$a_n - \nu^n = a_{n-1}$
2	1	$ia_{n-1} = 1 - \nu^{n-1}$	ν^{n-1}	$a_{n-1} - \nu^{n-1} = a_{n-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	1	$ia_{n-t+1} = 1 - \nu^{n-t+1}$	ν^{n-t+1}	$a_{n-t+1} - \nu^{n-t+1} = a_{n-t}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n-1	1	$ia_2 = 1 - \nu^2$	ν^2	$a_2 - \nu^2 = a_1$
n	1	$ia_1 = 1 - \nu$	ν	$a_1 - \nu = 0$
Total	n	$n - a_n$	a_n	

Clearly there is no amortization table for a perpetuity

- The original outstanding balance is $\frac{1}{i}$.
- The payment amount is 1.
- The interest pais is $\frac{1}{i} \cdot i = 1$.
- The principal repaid is $1 - 1 = 0$.

2.2 Observations

2.2.1 The outstanding balance is the balance according to the prospective method

2.2.2 The amount of principal repaid forms a geometric progression with ratio $(1+i)$.

May 2001 Exam Problem 31 Seth borrows X for four years at an annual effective interest rate of 8%, to be repaid with equal payments at the end of each year. The outstanding loan balance at the end of the second year is 1076.82 and at the end of the third year is 559.12 . Calculate the principal repaid in the first payment.

Solution: The principal reapid during the third year is

$$1076.82 - 559.12 = 517.70 \tag{35}$$

Since the principal repaid forms a geometric progression, the amount repaid during the first year is

$$\frac{517.70}{1.08^2} = 443.84 \tag{36}$$

May 2005 Exam Problem 25 A bank customer takes out a loan of 500 with a 16% nominal interest rate convertible quarterly. The customer makes payments of 20 at the end of each quarter. Calculate the amount of principal in the fourth payment.

Solution: The amount of interest paid during time period 1 is

$$500 \cdot 0.04 = 20.0 \tag{37}$$

Therefore, there is no interest repaid. Thus we also have that there is no interest repaid in any succeeding payment either. Notice that the value of a perpetity that pays 20 is

$$20 \cdot \frac{1}{0.04} = 500 \tag{38}$$

November 2005 Exam Problem 18 A loan is repaid with level annual payments based on an annual effective interest rate of 7%. The 8th payment consists of 789 of interest and 211 of principal. Calculate the amount of interest paid in the 18th payment.

Solution:

$$P = 789 + 211 \quad (39)$$
$$= 1000 \quad (40)$$

The principal repaid in the 18th payment is

$$211 \cdot 1.07^{10} = 415.07 \quad (41)$$

Therefore, the amount of principal repaid is

$$1000 - 415.07 = 584.93 \quad (42)$$

2.2.3 The sum of the principal payments is the original loan balance

Problem The aount of principal repaid in the 3^{rd} payment of a 5-year loan at 9% is 300. What is the original loan value?

Solution:

$$L_0 = P\nu^5 + P\nu^4 + P\nu^3 + P\nu^2 + P\nu \quad (43)$$
$$= P\nu^5 + P\nu^4 + 300 + P\nu^2 + P\nu \quad (44)$$
$$= \frac{300}{1.09^2} + \frac{300}{1.09} + 300 + 300(1.09) + 300(1.09)^2 \quad (45)$$
$$= \frac{300}{1.09^2} \cdot s_5 \quad (46)$$
$$= 252.5040 \cdot 5.9847 \quad (47)$$
$$= 1511.16 \quad (48)$$

2.2.4 The sum of the interest payments is equal to the difference between the sum of the total payments and the principal payments $k - a_k$

November 2000 Exam Problem 12 Kevin takes out a 10-year loan of L, which he repays by the amortization method at an annual effective interest rate of i. Kevin makes payments of 1000 at the end of each year . The total amount of interest repaid during the life of the loan is also equal to L. Calculate the amount of interest repaid during the first year of the loan.

Solution:

$$\begin{aligned} L &= 1000a_{10} && (49) \\ L &= 1000\,(10 - a_{10}) && (50) \\ 1000a_{10} &= 10000 - 1000a_{10} && (51) \\ a_{10} &= 5 && (52) \\ i &= 15.0984\% && (53) \\ I_1 &= 1000 \cdot a_{10} \cdot i && (54) \\ &= 1000 \cdot 5 \cdot .150984 && (55) \\ &= 755 && (56) \end{aligned}$$

3 Sinking Funds

Rather than repay a loan in installments by the amortization method, a borrower may choose to repay it by paying the interest as it accrues on the loan, and a lump sum payment at the end

1. The interest payment is $L_0 \cdot i$. Using our recursion relationship, we have

$$\begin{aligned} L_1 &= L_0(1+i) - P && (57) \\ &= L_0(1+i) - L_0 \cdot i && (58) \\ &= L_0 && (59) \end{aligned}$$

   Therefore, the lump sum amount that needs to be paid of at the end of the loan

2. Since we need to accumulate L_0 in the side-fund at $t = n$, so the interest rate credited to the side fund is j, the requred deposit is $\frac{L}{s_n}$.

For insatnce, consider a loan for 1000 at an annual effective rate of 5% for 10 years.

1. The interest payment would be $1000 \cdot .05 = 50$

2. The deposit to the side fund would be $\frac{1000}{s_{10}} = \frac{1000}{12.5779} = 79.50$

3. The total annual payment would be $50.00 + 79.50 = \$129.50$

4. If the loan would be repaid by the normal amortization method, the annual payment would be $\frac{1000}{a_{10}} = \frac{1000}{7.7217} = 129.50$

This will be always be the case if the interest rate on the loan and the side fund are the same. This can be show as follows:

$$L \cdot i + \frac{L}{s_n} = L\left(i + \frac{i}{(1+i)^n - 1}\right) \tag{60}$$

$$= Li\left(\frac{(1+i)^n - 1 + 1}{(1+i)^n - 1}\right) \tag{61}$$

$$= L\left(\frac{i}{1 - \nu^n}\right) \tag{62}$$

$$= \frac{L}{a_n} \tag{63}$$

If the two rates are not the same, then

$$\frac{1}{a_{n|i\&j}} = \frac{1}{s_{n|j}} + i \tag{64}$$

$$= \left(\frac{1}{a_{n|j}} - j\right) + i \tag{65}$$

$$= \frac{1 + (i-j)a_{n|j}}{a_{n|j}} \tag{66}$$

$$a_{n|i\&j} = \frac{a_{n|j}}{1 + (i-j)a_{n|j}} \tag{67}$$

May 2005 Exam Problem 2 Lori borrows 10,000 for 10 years at an annual effective interest rate of 9%. At the end of each year, she pays the interest on the loan and deposits the level amount necessary to repay the principal to a sinking fund earning an annual effective interest rate of 8%. The total payments made by Lori over the 10-year period is X. Calculate X.

Solution:

$$P = 10000 \cdot \left[\frac{1}{s_{10|0.08}} + 0.09\right] \tag{68}$$
$$= 10000 \cdot \left[\frac{1}{14.4866} + 0.09\right] \tag{69}$$
$$= 10000 \cdot .159029 \tag{70}$$
$$= 1590.23 \tag{71}$$
$$X = 10P \tag{72}$$
$$= 15902.93 \tag{73}$$

May 2001 Exam Problem 4 A 20-year loan of 20,000 may be repaid under the following two methods:

1. The amortization method with equal annual payments at an annual effective rate of 6.5%.
2. The sinking fund method in which the lender receives an annual effective rate of 8% and the sinking fund earns an annual effective rate of j.

Both methods require a payment of X to be made at the end of each year for 20 years. Calculate j.

Solution:

$$X = \frac{20000}{a_{20|0.65}} \tag{74}$$
$$= \frac{20000}{11.0185} \tag{75}$$
$$= 1815.13 \tag{76}$$
$$1815.13 = \frac{20000}{s_{20|j}} + 20000 \cdot 0.08 \tag{77}$$
$$215.13 = \frac{20000}{s_{20|j}} \tag{78}$$
$$s_{20|j} = 92.9670 \tag{79}$$
$$j = 14.179\% \tag{80}$$

We can determine the amortization schedule for the sinking-fund method where the interest on the side fund is the same as the loan as follows:

$$L_1 = L_0(1+i) - P \tag{81}$$
$$= L_0 + L_0 \cdot i - \left(L_0 \cdot i + \frac{L_0}{s_n}\right) \tag{82}$$
$$= L_0 - \frac{L_0}{s_n} \tag{83}$$
$$= L_0 - \frac{L_0}{s_n} \cdot s_1 \tag{84}$$
$$L_2 = L_1(1+i) - P \tag{85}$$
$$= \left(L_0 - \frac{L_0}{s_n}\right)(1+i) - \left(L_0 \cdot i + \frac{L_0}{s_n}\right) \tag{86}$$
$$= L_0 - \frac{L_0}{s_n}\left[(1+i)+1\right] \tag{87}$$
$$= L_0 - \frac{L_0}{s_n} \cdot s_2 \tag{88}$$
$$= L_0\left[1 - \frac{s_2}{s_n}\right] \tag{89}$$
$$= L_0 \cdot \frac{s_n - s_2}{s_n} \tag{90}$$
$$= L_0 \cdot \frac{(1+i)^2 s_{n-2}}{s_n} \tag{91}$$
$$= L_0 \cdot \frac{(1+i)^n\, a_{n-2}}{(1+i)^n\, a_n} \tag{92}$$
$$= \frac{L_0}{a_n} \cdot a_{n-2} \tag{93}$$
$$= P a_{n-2} \tag{94}$$

Therfore, the outstanding balance at time $t = k$

1. Under the sinfing-fund method is $L_0 - \frac{L_0}{s_n} \cdot s_k$
2. Under the amortization method is $\frac{L_0}{a_n} \cdot a_{n-k}$
3. The two are equivalent.

Problem A loan of 100,000 is to be repaid by 20 level annual payments. The lender wishes to earn 12% on the full loan amount and wil deposit the remainder of the annual payment to a sinking fund earning 8% annually.

1. Find the amount of the loan.

$$100,000 \cdot 0.12 + \frac{100,000}{s_{20|.08}} = 12,000 + \frac{100,000}{45.7620} \quad (95)$$

$$= 12,000.00 + 2,185.22 \quad = 14,185.22 \quad (96)$$

2. Just after receiving the 10^{th} payment, the lender sells the remaining 10 payments. The purchaser considers two ways of valuing the remaining payments:

   (a) amortization at 10%, or

   (b) earning an annual return of 12% on his investment while recovering his principal in a sinking fund earning 8%.

   Find the amount of the original lender's sinking fund at the time the remainder of the loan is sold, and in each case find the amount paid by the investor to the original lender.

   Solution: The amount in the sinking fund is

$$2,185.22 s_{10} = 2,185.22 \cdot 14.4866 \quad (97)$$

$$= 31,656.41 \quad (98)$$

   The remaining 10 payments have a value using the amortization method and 10% of

$$14,185.22 a_{10|.10} = 14,185.22 \cdot 6.1446 \quad (99)$$

$$= 87,162.50 \quad (100)$$

   Based on the alternative metod

$$14,185.22 = .12 \cdot L_0 + \frac{L_0}{s_{10|.08}} \quad (101)$$

$$= .12 \cdot L_0 + \frac{L_0}{14.4866} \quad (102)$$

$$= .12 \cdot L_0 + .0690 \cdot L_0 \quad (103)$$

$$= .1890 \cdot L_0 \quad (104)$$

$$L_0 = 75,054.07 \quad (105)$$

4 Varying Series of Payments

Consider a loan L to be repaid with n periodic payments $R_1, R_2, \ldots, R_n$. The equation of value is

$$L = \sum_{t=1}^{n} \nu^t R_t \quad (106)$$

At this point we have considered level payments with the additional possibility of a balloon or drop payment at the end. We now consider other possible scenarios.

May 1984 CAS Exam Problem 8 A loan is repaid with 20 increasing annual installments of 1,2,3,. . . ,20. The payments begin one year after the loan is made. Find the principal contained in the 10^th payment, if the annual interest rate is 4%.

Solution: the outstanding balance at $t = 9$ by the prospective method is

$$\begin{aligned} L_9 &= 10a_{11} + (Ia)_{11} && (107)\\ &= 9 \cdot 8.7605 + \frac{\ddot{a}_{11} - 11\nu^{11}}{.04} && (108)\\ &= 78.84 + 49.14 && (109)\\ &= 127.98 && (110) \end{aligned}$$

The principal repaid will be the payment minus the interest on the outstanding balance.

$$\begin{aligned} PR &= 10 - 127.98 \cdot 0.04 && (111)\\ &= 4.88 && (112) \end{aligned}$$

Problem A loan of 3000 at an effective quarterly interest rate of $j = .02$ is amortized by means of 12 quarterly payments , beginning one quarter after the loan is made. Each payment consists of a principal repayment of 250 plus interest due on the previous quarter's outstanding balance. Construct the amortization schedule.

Solution:

t	Payment	Interest Due	Principal Repaid	Outstanding Balance
0				3000
1	310	60	250	2750
2	305	55	250	2500
3	300	50	250	2250
⋮	⋮	⋮	⋮	⋮
11	260	10	250	250
12	255	5	250	0

We see from the amortization table that this loan is equivalent to

$$\begin{aligned} L &= 250a_{12|.02} + 5(Da)_{12|.02} && (113)\\ &= 250 \cdot 10.5753 + 5\,\frac{12 - 10.5753}{.02} && (114)\\ &= 2643.83 + 356.17 && (115)\\ &= 3000.00 && (116)\end{aligned}$$

Problem In order to repay a school loan, a payment schedule of 200 at the end of the year for the first 5 years, 1200 at the end of the year for the next 5 years, and 2200 at the end of the year for the final 5 years is agreed upon.

1. If interest is at the annual effective rate of $i = 6\%$, what is the loan value?

$$\begin{aligned} L &= 2200a_{15} - 1000a_{10} - 1000a_5 && (117)\\ &= 21366.95 - 7360.09 - 4212.36 && (118)\\ &= 9794.50 && (119)\end{aligned}$$

2. Construct the amortization scedule for the loan for the first 7 years.

t	Payment	Interest Due	Principal Repaid	Outstanding Balance
0				9794.50
1	200.00	587.67	-387.67	10182.17
2	200.00	610.93	-410.93	10593.10
3	200.00	635.59	-435.59	11028.69
4	200.00	661.72	-461.72	11490.41
5	200.00	689.42	-489.42	11979.83
6	1200.00	718.79	481.21	11498.62
7	1200.00	689.92	510.08	10988.54

Notice that the during the first 5 years of the loan repayment, the principal repaid is negative (yet, it still forms a geometric progression).

3. Verify the outstanding balance of the loan at the end of the 7^{th} year by the prospective method.

$$\begin{aligned} OB &= 2200a_8 - 1000a_3 && (120) \\ &= 13661.55 - 2673.01 && (121) \\ &= 10988.54 && (122) \end{aligned}$$