Chapter 10

Roots and Radicals

Section 10.1 – Finding Square Roots

We know that if

$$x = 5$$
$$x^2 = 25$$

Also, if

$$x = -5$$
$$x^2 = 25$$

Section 10.1 – Finding Square Roots

We want to concern ourselves with the reverse problem. If

$$x^2 = 25$$

$$x = ?$$

Section 10.1 – Finding Square Roots

Problem:

Find all square roots of 64.

Since

$$8 \times 8 = 64$$

$$(-8)(-8) = 64$$

The square root of 64 is 8 and -8.

Section 10.1 – Finding Square Roots

We write the positive square root of 64 as

$$\sqrt{64} = 8$$

We write the negative square root of 64 as

$$-\sqrt{64} = 8$$

Section 10.1 – Finding Square Roots

$\sqrt{}$ is called the radical sign

$\sqrt{64}$ 64 as is called the radicand

$\sqrt{64}$ The entire expression is called a radical

Section 10.1 – Finding Square Roots

Problem:

Find $-\sqrt{225}$

This represents the negative square root of 225.

Since $15^2 = 225$

$$-\sqrt{225} = -15$$

Section 10.1 – Finding Square Roots

<u>**Problem**</u>:

Find $$\sqrt{\frac{9}{16}}$$

Since $$\left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$\sqrt{\frac{9}{16}} = \frac{3}{4}$$

Section 10.1 – Finding Square Roots

We have shown that

$$\sqrt{64} = 8$$

Then

$$\sqrt{64} \times \sqrt{64} = 8 \times 8$$

or

$$\left(\sqrt{64}\right)^2 = 64$$

Section 10.1 – Finding Square Roots

Problem:

$$\left(\sqrt{2x^2 + 3}\right)^2$$

$$\left(\sqrt{2x^2 + 3}\right)^2 = 2x^2 + 3$$

Section 10.1 – Rational, Irrational, or not a Real Number

So far we have only considered rational numbers whose square roots are themselves a rational number. We call these numbers perfect squares.

Examples of perfect squares are 1, 4, 9, 16, 25 and ¼.

Section 10.1 – Rational, Irrational, or not a Real Number

But the square root of 7 is not a rational number since we can not find integers p and q such that

$$\left(\frac{p}{q}\right)^2 = 7$$

We call the square root of 7 an **irrational number**.

Section 10.1 – Rational, Irrational, or not a Real Number

To evaluate the square root of 7, you can use the square root key on your calculator.

$$\sqrt{7} \approx 2.645751331$$

$\approx$ means "is approximately equal to"

Section 10.1 – Rational, Irrational, or not a Real Number

The square root of a negative number is neither rational or irrational. It is **not** a real number!

Example

$\sqrt{-100}$ is not a real number.

However,

$$-\sqrt{100} = -10$$

Section 10.1 – Finding Higher Roots

Finding higher orders of roots is similar.

If

$$x = 3$$

$$x^3 = 27$$

Then

$$\sqrt[3]{27} = 3$$

Section 10.1 – Finding Higher Roots

Furthermore

$$x = -3$$

$$x^3 = -27$$

Then

$$\sqrt[3]{-27} = -3$$

Section 10.1 – Finding Higher Roots

Observation

$$\sqrt[3]{-27} = -3$$

But

$$\sqrt{-27}$$ is not a real number.

Section 10.1 – Finding Higher Roots

Also,

$$x = 3$$

$$x^4 = 81$$

Then

$$\sqrt[4]{81} = 3$$

Section 10.1 – Finding n^{th} Roots of n^{th} Powers

We have seen that

$$\sqrt{15^2} = \sqrt{225} = 15$$

$$\sqrt{(-15)^2} = \sqrt{225} = 15$$

Section 10.1 – Finding n^th Roots of n^th Powers

Recall that

$$|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

So we have that

$$|15| = 15$$

$$|-15| = -(-15) = 15$$

Section 10.1 – Finding n^{th} Roots of n^{th} Powers

Therefore, for any real number a

$$\sqrt{a^2} = |a|$$

Moreover, for any even positive integer n and real number a

$$\sqrt[n]{a^n} = |a|$$

Section 10.1 – Finding nth Roots of nth Powers

However, for any odd positive integer n and real number a

$$\sqrt[n]{a^n} = a$$

For instance,

$$\sqrt[3]{5^3} = \sqrt[3]{125} = 5$$

$$\sqrt[3]{(-5)^3} = \sqrt[3]{-125} - 5$$

Section 10.2 – Rational Exponents

We know that for integer exponents

$$x^n \cdot x^m = x^{n+m}$$

For instance

$$x^2 \cdot x^3 = x^{2+3} = x^5$$

Section 10.2 – Rational Exponents

What happens if the exponents are fractions instead?

$$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x$$

If x=25

$$25^{\frac{1}{2}} \cdot 25^{\frac{1}{2}} = 25^{\frac{1}{2}+\frac{1}{2}} = 25$$

Section 10.2 – Rational Exponents

But we also know that

$$\sqrt{25} \cdot \sqrt{25} = 25$$

Therefore, we must have that

$$25^{\frac{1}{2}} = \sqrt{25}$$

Section 10.2 – Rational Exponents

What's more

$$x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x$$

So we also have that

$$x^{\frac{1}{3}} = \sqrt[3]{x}$$

Section 10.2 – Rational Exponents

Observation

$$x^{\frac{1}{m}} = \sqrt[m]{x}$$

Section 10.2 – Rational Exponents

Examples

$$32^{\frac{1}{5}} = 5$$

$$(-64)^{\frac{1}{3}} = -4$$

$$(-64)^{\frac{1}{4}} = \text{Not a real}$$

Section 10.2 – Rational Exponents

Similarly,

$$27^{\frac{2}{3}} = 27^{\frac{1}{3}+\frac{1}{3}}$$

$$= \left(27^{\frac{1}{3}}\right)^2$$

$$= 3^2$$

$$= 9$$

Section 10.2 – Rational Exponents

Or in general,

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{m}} \right)^n$$

$$= \left(\sqrt[m]{x} \right)^n$$

Section 10.2 – Rational Exponents

Examples

$$25^{\frac{3}{2}} = \left(25^{\frac{1}{2}}\right)^3 = 5^3 = 125$$

$$(-32)^{\frac{3}{5}} = \left([-32]^{\frac{1}{5}}\right)^3 = (-2)^3 = -8$$

$$-32^{\frac{3}{5}} = -\left(32^{\frac{1}{5}}\right)^3 = -2^3 = -8$$

Section 10.2 – Rational Exponents

We can extend this rule to negative exponents. Recall that

$$7^{-2} = \frac{1}{7^2} = \frac{1}{49}$$

We extend this to rational exponents

$$49^{-\frac{3}{2}} = \left(49^{\frac{3}{2}}\right)^{-1} = \frac{1}{49^{\frac{3}{2}}} = \frac{1}{\left(49^{\frac{1}{2}}\right)^{3}} = \frac{1}{343}$$

Section 10.2 – Rational Exponents

Examples

$$\left(-\frac{64}{125}\right)^{-\frac{2}{3}} = \frac{1}{\left(-\frac{64}{125}\right)^{\frac{2}{3}}} = \frac{1}{\left(\sqrt[3]{-\frac{64}{125}}\right)^{2}}$$

$$= \frac{1}{\left(-\frac{4}{5}\right)^{2}} = \frac{1}{\frac{16}{25}} = \frac{25}{16}$$

Section 10.2 – Rational Exponents

Examples

$$\frac{\left(x^{\frac{2}{3}}\right)^{2}}{\left(x^{2}\right)^{\frac{7}{3}}} = \frac{x^{\frac{4}{3}}}{x^{\frac{14}{3}}} = \frac{1}{x^{\frac{14}{3}-\frac{4}{3}}} = \frac{1}{x^{\frac{10}{3}}}$$

Section 10.2 – Rational Exponents

Examples

$$\sqrt[3]{xz} \cdot \sqrt{z} = (xz)^{\frac{1}{3}} \cdot z^{\frac{1}{2}}$$

$$= x^{\frac{1}{3}} \cdot z^{\frac{1}{3}} \cdot z^{\frac{1}{2}}$$

$$= x^{\frac{1}{3}} \cdot z^{\frac{1}{3}+\frac{1}{2}}$$

$$= x^{\frac{1}{3}} \cdot z^{\frac{5}{6}}$$

Section 10.3 – Simplifying Radical Expressions

In this section, we want to learn how to multiply and divide algebraic expressions that contain radicals.

Section 10.3 – Use the Product Rule

$$\sqrt{16} \cdot \sqrt{25} = 4 \cdot 5 = 20$$

$$\sqrt{16 \cdot 25} = \sqrt{400} = 20$$

Therefore,

$$\sqrt{16} \cdot \sqrt{25} = \sqrt{16 \cdot 25}$$

Section 10.3 – Use the Product Rule

For nonnegative real numbers x and y,

$$\sqrt{x} \cdot \sqrt{y} = \sqrt{x \cdot y} \text{ and } \sqrt{x \cdot y} = \sqrt{x} \cdot \sqrt{y}$$

That is, the product of two radicals is the radical of the product.

We can use this rule to simplify expressions with radicals.

Section 10.3 – Use the Product Rule

Problem:

Simplify $\sqrt{60}$

$$60 = 2^2 \cdot 3 \cdot 5$$

$$\sqrt{60} = \sqrt{2^2 \cdot 3 \cdot 5}$$

$$= \sqrt{2^2} \cdot \sqrt{3 \cdot 5}$$

$$= 2 \cdot \sqrt{15}$$

Section 10.3 – Use the Product Rule

Problem:

Simplify $\sqrt{10} \cdot \sqrt{50}$

$$\sqrt{10} \cdot \sqrt{50} = \sqrt{500}$$

$$500 = 2^2 \cdot 5^3 = 2^2 \cdot 5^2 \cdot 5 = (2 \cdot 5)^2 \cdot 5$$

$$\sqrt{500} = \sqrt{10^2 \cdot 5}$$

$$= \sqrt{10^2} \cdot \sqrt{5}$$

$$= 10 \cdot \sqrt{5}$$

Section 10.3 – Use the Quotient Rule

By the product rule we have

$$\sqrt{y} \cdot \sqrt{\frac{x}{y}} = \sqrt{y \cdot \frac{x}{y}}$$

$$= \sqrt{x}$$

Therefore,

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

Section 10.3 – Use the Quotient Rule

<u>Problem</u>:
Simplify $\sqrt{\frac{10}{49}}$

$$\frac{10}{49} = \frac{2 \cdot 5}{7^2}$$

$$\sqrt{\frac{10}{49}} = \frac{\sqrt{10}}{\sqrt{7^2}}$$

$$= \frac{\sqrt{10}}{7}$$

Section 10.3 – Simplifying Radical Expressions

The product and quotients rules work for radicals of different powers as well.

$$\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{x \cdot y}$$

$$\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}$$

Section 10.3 – Simplifying Radical Expressions

Problem:
Simplify $\sqrt[3]{-\dfrac{27}{64}}$

$$\sqrt[3]{-\frac{27}{64}} = \frac{\sqrt[3]{-27}}{\sqrt[3]{64}} = \frac{\sqrt[3]{(-3)^3}}{\sqrt[3]{4^3}} = -\frac{3}{4}$$

Section 10.3 – Simplified Radicals

Simplified Radical

2. The radicand has no factor raised to a power greater than or equal to the index
3. The radicand has no fractions.
4. No denominator contains a radical.
5. Exponents in the radicand and the index of the radical have no common factors.

Section 10.3 – Simplifying Radicals

<u>Problem</u>:

Simplify $\dfrac{8\sqrt{50}}{4\sqrt{5}}$

$$\frac{8\sqrt{50}}{4\sqrt{5}} = \frac{8}{4}\cdot\frac{\sqrt{50}}{\sqrt{5}} = 2\cdot\sqrt{\frac{50}{5}} = 2\cdot\sqrt{10}$$

Section 10.3 – Simplifying Radicals

Problem:
Simplify $\frac{\sqrt{3}}{\sqrt{8}} \cdot \sqrt{\frac{7}{2}}$

$$\frac{\sqrt{3}}{\sqrt{8}} \cdot \sqrt{\frac{7}{2}} = \frac{\sqrt{3 \cdot 7}}{\sqrt{8 \cdot 2}} = \frac{\sqrt{3 \cdot 7}}{\sqrt{2^4}} = \frac{\sqrt{21}}{\sqrt{(2^2)^2}} = \frac{\sqrt{21}}{4}$$

Section 10.3 – Simplifying Radicals

All the rules we discussed so far apply to variables as well as long as the variables represent nonnegative numbers.

$$\sqrt{(-7)^2} \neq -7$$

Therefore, if p=-7

$$\sqrt{(p)^2} \neq p$$

Section 10.3 – Simplifying Radicals

Problem:

Simplify $\sqrt{100p^8}$ Assume p>0

$$\sqrt{100p^8} = \sqrt{10^2 \cdot \left(p^4\right)^2}$$

$$= \sqrt{10^2} \cdot \sqrt{\left(p^4\right)^2}$$

$$= 10p^4$$

Section 10.3 – Pythagorean Formula

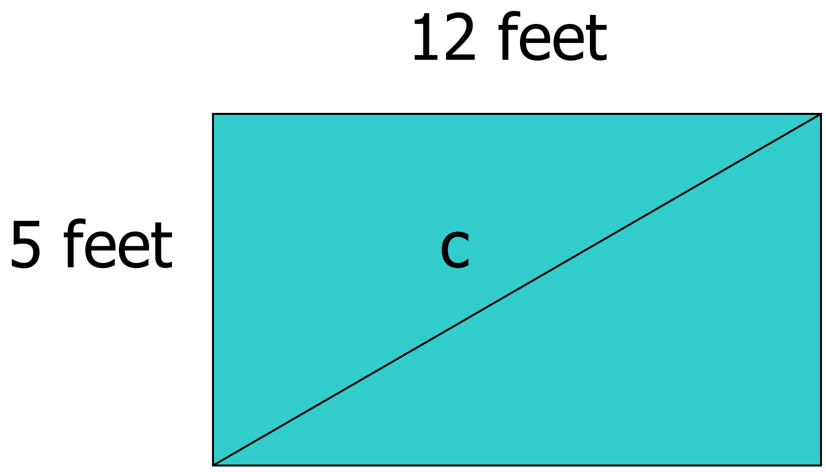
Problem

A rectangle has dimension 5 feet by 12 feet. Find the length of its diagonal.

Section 10.3 – Pythagorean Formula

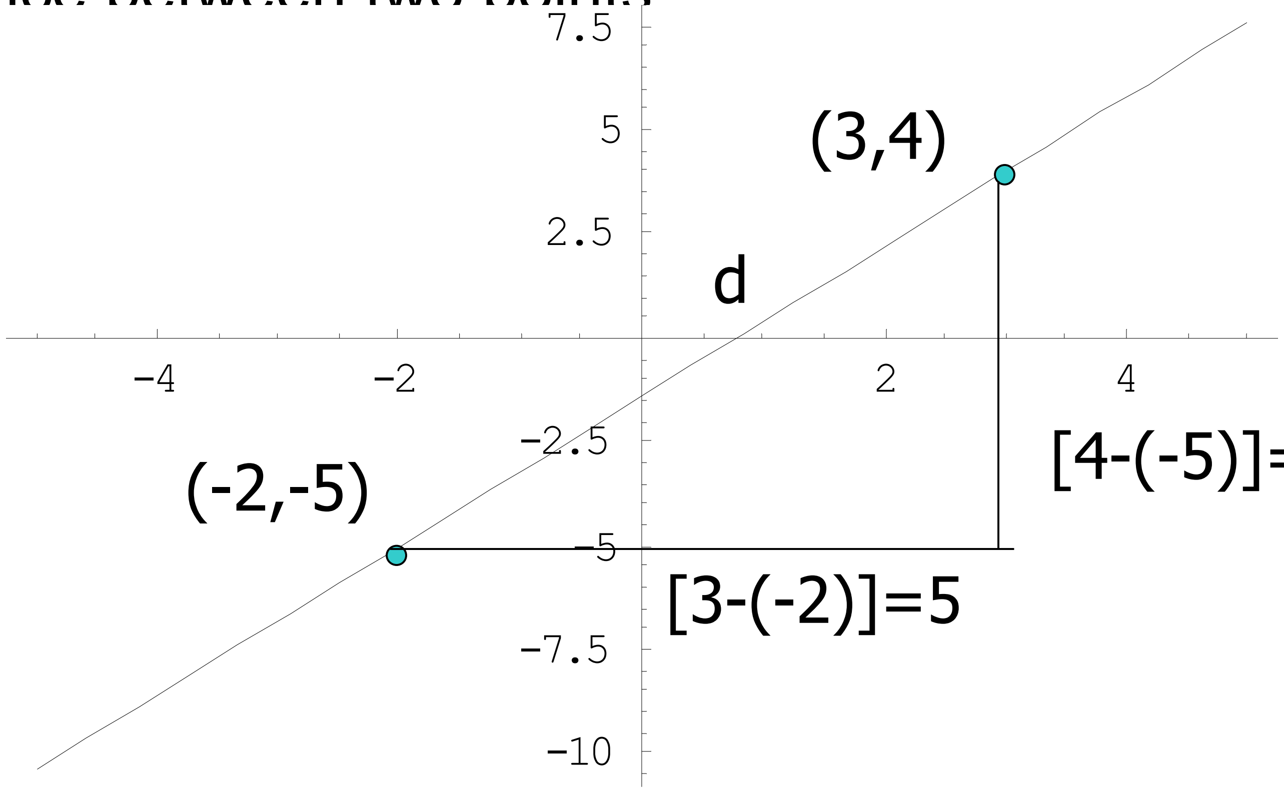
By the Pythagorean Formula

$$c^2 = 5^2 + 12^2$$
$$= 169$$
$$c = \sqrt{169}$$
$$= 13$$



Section 10.3 – Pythagorean Formula

We can use the Pythagorean Formula to find the distance between two points



Section 10.3 – Pythagorean Formula

By the Pythagorean Formula

$$d = \sqrt{5^2 + 9^2}$$

$$= \sqrt{106}$$

$$\approx 10.2956$$

Section 10.3 – Pythagorean Formula

In general, the distance between two points (x_1,y_1) and (x_2,y_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Section 10.4 – Addition and Subtraction of Radicals

Like radicals are terms that have multiples of the same root of the same radicand.

$\sqrt{7}$ and $\sqrt{11}$ Different radicands

$\sqrt[3]{7}$ and $\sqrt{7}$ Different root

$2\sqrt{7}$ and $5\sqrt{7}$ Bingo!

Section 10.4 – Addition and Subtraction of Radicals

<u>Problem</u>:

Add $-4\sqrt{3}+9\sqrt{3}$

By the distributive law

$$-4\sqrt{3}+9\sqrt{3}=(-4+9)\sqrt{3}$$

$$=5\sqrt{3}$$

Section 10.4 – Addition and Subtraction of Radicals

Problem:

Add $\sqrt[3]{11} + 7\sqrt[3]{11}$

By the distributive law

$$\sqrt[3]{11} + 7\sqrt[3]{11} = (1 + 7)\sqrt[3]{11}$$

$$= 8\sqrt[3]{11}$$

Section 10.4 – Addition and Subtraction of Radicals

Problem:

Simplify $5\sqrt{200} - 6\sqrt{18}$

$$5\sqrt{200} - 6\sqrt{18} = 5\sqrt{2 \cdot 2^2 \cdot 5^2} - 6\sqrt{2 \cdot 3^2}$$

$$= 5 \cdot 2 \cdot 5\sqrt{2} - 6 \cdot 3\sqrt{2}$$

$$= 50\sqrt{2} - 18\sqrt{2}$$

$$= 32\sqrt{2}$$

Section 10.4 – Addition and Subtraction of Radicals

Important!

$$\sqrt{2} + \sqrt{8} \neq \sqrt{10}$$

Section 10.4 – Addition and Subtraction of Radicals

Problem:

Simplify $\sqrt{3r}\cdot\sqrt{6}+\sqrt{8r}$

$$\sqrt{3r}\cdot\sqrt{6}+\sqrt{8r} = \sqrt{18r}+\sqrt{8r}$$

$$= \sqrt{2\cdot 3^2 r}+\sqrt{2\cdot 2^2 r}$$

$$= 3\sqrt{2r}+2\sqrt{2r}$$

$$= 5\sqrt{2r}$$

Section 10.4 – Addition and Subtraction of Radicals

Problem:
Simplify

$$\sqrt{\frac{8}{9}} + \sqrt{\frac{18}{36}} = \frac{\sqrt{2 \cdot 4}}{\sqrt{9}} + \frac{\sqrt{2 \cdot 9}}{\sqrt{36}}$$

$$= \frac{2\sqrt{2}}{3} + \frac{3\sqrt{2}}{6} = \frac{4\sqrt{2}}{6} + \frac{3\sqrt{2}}{6}$$

$$= \frac{7\sqrt{2}}{6}$$

Section 10.4 – Addition and Subtraction of Radicals

Problem:
Simplify

$$\sqrt{\frac{25}{x^8}} - \sqrt{\frac{9}{x^6}} = \frac{\sqrt{25}}{\sqrt{x^8}} - \frac{\sqrt{9}}{\sqrt{x^6}}$$

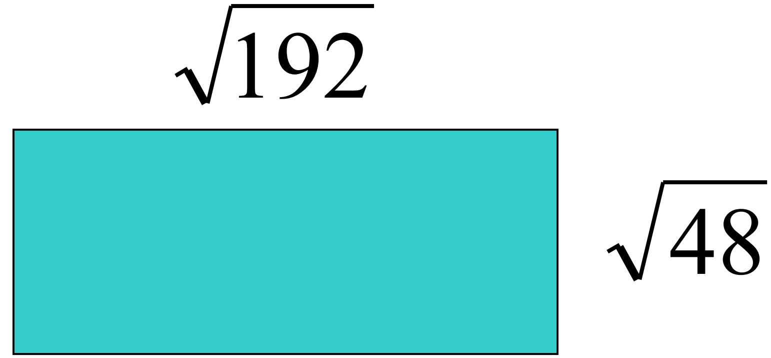
$$= \frac{5}{x^4} - \frac{3}{x^3} = \frac{5}{x^4} - \frac{3x}{x^4}$$

$$= \frac{5 - 3x}{x^4}$$

Section 10.4 – Addition and Subtraction of Radicals

Problem:

Find the perimeter of the rectangle



$$P = 2\left(\sqrt{192} + \sqrt{48}\right)$$

$$P = 2\left(8\sqrt{3} + 4\sqrt{3}\right)$$

$$P = 24\sqrt{3}$$

Section 10.5 – Multiplying Radical Expressions

Problem:

Find the product and simplify

$$(\sqrt{2} + 5\sqrt{3})(\sqrt{3} - 2\sqrt{2})$$

Section 10.5 – Multiplying Radical Expressions

$$(\sqrt{2} + 5\sqrt{3})(\sqrt{3} - 2\sqrt{2})$$

$$= \sqrt{2}\sqrt{3} - 2\sqrt{2}\sqrt{2} + 5\sqrt{3}\sqrt{3} - 10\sqrt{3}\sqrt{2}$$ FOIL

$$= \sqrt{6} - 2 \cdot 2 + 5 \cdot 3 - 10\sqrt{6}$$ Product rule

$$= -4 + 15 - 9\sqrt{6}$$ Combine like radicals

$$= 11 - 9\sqrt{6}$$

Section 10.5 – Multiplying Radical Expressions

Problem:

Find the product and simplify

$$(\sqrt{3} - 2)(\sqrt{3} + 2)$$

Section 10.5 – Multiplying Radical Expressions

$$(\sqrt{3} - 2)(\sqrt{3} + 2)$$

$$= \left(\sqrt{3}\right)^2 - 2^2$$

Special formula for (x+y)(x-y)

$$= 3 - 4$$

$$\left(\sqrt{3}\right)^2 = 3$$

$$= -1$$

Section 10.5 – Multiplying Radical Expressions

The pair of factors like those in the previous problem

$$(\sqrt{3} - 2)(\sqrt{3} + 2) = -1$$

are called **conjugates** of each other. Notice that their product produced a rational number. A conjugate of a factor can be used to simplify the denominator of an expression.

Section 10.5 – Multiplying Radical Expressions

Problem:

Find the product and simplify

$$(\sqrt{13} - 2)^2$$

Section 10.5 – Multiplying Radical Expressions

$$(\sqrt{13} - 2)^2$$

$$= \left(\sqrt{13}\right)^2 - 2 \cdot \sqrt{12} \cdot 2 + 2^2$$ Special formula for $(x-y)^2$

$$= 13 - 4\sqrt{13} + 4$$

$$= 14 - 4\sqrt{13}$$

Section 10.5 – Multiplying Radical Expressions

Problem:

Find the product and simplify

$$(4+\sqrt[3]{7})(4-\sqrt[3]{7})$$

Section 10.5 – Multiplying Radical Expressions

$(4+\sqrt[3]{7})(4-\sqrt[3]{7})$

$= 16-\left(\sqrt[3]{7}\right)^2$ Special formula for (x+y)(x-y)

$= 16-\sqrt[3]{7^2}$

$= 16-\sqrt[3]{49}$

Section 10.5 – Rationalizing the Denominator

Problem:

Simplify by rationalizing the denominator.

$$\frac{5}{\sqrt{11}}$$

$$\frac{5}{\sqrt{11}} = \frac{5}{\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{11}} = \frac{5\sqrt{11}}{11}$$

Section 10.5 – Rationalizing the Denominator

Problem:

Simplify by rationalizing the denominator.

$$\sqrt{\frac{8}{45}}$$

Section 10.5 – Rationalizing the Denominator

$$\sqrt{\frac{8}{45}} = \frac{\sqrt{8}}{\sqrt{45}} \cdot \frac{\sqrt{45}}{\sqrt{45}}$$

$$= \frac{\sqrt{2 \cdot 2^2 \cdot 3^2 \cdot 5}}{45}$$

$$= \frac{6\sqrt{10}}{45}$$

$$= \frac{2\sqrt{10}}{15}$$

Section 10.5 – Rationalizing the Denominator

Problem:

Simplify by rationalizing the denominator.

$$\sqrt{\frac{200k^6}{y^7}}$$

Section 10.5 – Rationalizing the Denominator

$$\sqrt{\frac{200k^6}{y^7}} = \frac{\sqrt{2 \cdot 2^2 \cdot 5^2 \cdot k^6}}{\sqrt{y \cdot y^6}}$$

$$= \frac{10k^3\sqrt{2}}{y^3\sqrt{y}}$$

$$= \frac{10k^3\sqrt{2y}}{y^4}$$

Section 10.5 – Rationalizing the Denominator

Problem:

Simplify by rationalizing the denominator.

$$\sqrt[3]{\frac{15}{32}}$$

Section 10.5 – Rationalizing the Denominator

$$\sqrt[3]{\frac{15}{32}} = \frac{\sqrt[3]{15}}{\sqrt[3]{2^2 \cdot 2^3}}$$

$$= \frac{\sqrt[3]{15}}{2 \cdot \sqrt[3]{2^2}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}}$$

$$= \frac{\sqrt[3]{30}}{4}$$

Section 10.5 – Simplifying Radical Expressions

Problem:

Simplify by rationalizing the denominator.

$$\frac{3}{2-\sqrt{5}}$$

Section 10.5 – Simplifying Radical Expressions

$$\frac{3}{2-\sqrt{5}} = \frac{3}{2-\sqrt{5}} \cdot \frac{2+\sqrt{5}}{2+\sqrt{5}}$$

multiply by conjugate

$$= \frac{3\left(2+\sqrt{5}\right)}{2^2-\left(\sqrt{5}\right)^2}$$

(x+y)(x-y)=x²-y²

$$= \frac{3\left(2+\sqrt{5}\right)}{4-5}$$

$$\left(\sqrt{5}\right)^2 = 5$$

$$= -3\left(2+\sqrt{5}\right)$$

Section 10.5 – Simplifying Radical Expressions

Problem:

Simplify by rationalizing the denominator.

$$\frac{2}{\sqrt{k}+\sqrt{z}}$$

Section 10.5 – Simplifying Radical Expressions

$$\frac{2}{\sqrt{k}+\sqrt{z}} = \frac{2}{\sqrt{k}+\sqrt{z}} \cdot \frac{\sqrt{k}-\sqrt{z}}{\sqrt{k}-\sqrt{z}}$$

$$= \frac{2\left(\sqrt{k}-\sqrt{z}\right)}{k-z}$$