Math Problem of the Month: October 2007

If a pyramid whose base is the shape of an equilateral triangle is built out of billiard balls, and this pyramid has n layers/levels of balls, then how many balls does the entire pyramid contain?

Solution Consider $L_k :=$ # of balls in the kth layer. Then

$$\begin{aligned} L_1 &= 1 \\ L_2 &= 1+2 \\ L_3 &= 1+2+3 \\ &\vdots \\ L_k &= 1+2+3+\cdots+k \\ &\vdots \\ L_n &= 1+2+3+\cdots+\cdots+n \end{aligned}$$

Letting $T_n :=$ total # of balls in the pyramid, we see that

$$T_n = L_1 + L_2 + L_3 + \cdots + L_n \ .$$

Method 1 We note that above there are n occurences of the number 1, $n-1$ occurences of the number 2, $n-2$ occurences of the number 3, etc. So

$$T_n = n(1) + (n-1)2 + (n-2)(3) + \cdots + 2(n-1) + 1(n) \ .$$

Realizing that the factors in each term add up to $n+1$, we may write T_n in summation notation,

$$T_n = \sum_{k=1}^{n} (k)(n+1-k) \ .$$

Here we'll use some known formulas for the sum of the first n integers and the sum of the first n squares.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \ , \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \ .$$

Once you know these formulas, they may be proven by "induction". Finding them is the hard part. They are in any calculus text although they do not belong to the subject of calculus!

The index of summation is k, so n is a constant which will factor out of the sums. We therefore have

$$\begin{aligned} T_n &= \sum_{k=1}^{n} (k)(n+1-k) \\ &= \sum_{k=1}^{n} (k)(n+1) - \sum_{k=1}^{n} k^2 \end{aligned}$$

$$
\begin{aligned}
&= (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 \\
&= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} \\
&= (n+1)\left(\frac{3n(n+1)}{6} - \frac{n(2n+1)}{6}\right) \\
&= (n+1)\left(\frac{n^2+2n}{6}\right) \\
&= \frac{n(n+1)(n+2)}{6}\ .
\end{aligned}
$$

Method 2 We'll still use the summation formulas. We realize that

$$L_k = \sum_{j=1}^{k} j\ .$$

so

$$
\begin{aligned}
T_n &= L_1 + L_2 + L_3 + \cdots + L_n \\
&= \sum_{k=1}^{n} L_k \\
&= \sum_{k=1}^{n}\sum_{j=1}^{k} j \\
&= \sum_{k=1}^{n} \frac{k(k+1)}{2} \\
&= \sum_{k=1}^{n} \frac{k^2}{2} + \sum_{k=1}^{n} \frac{k}{2} \\
&= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right) \\
&= \frac{n+1}{2}\left(\frac{n(2n+1)}{6} + \frac{n}{2}\right) \\
&= \frac{n+1}{2}\left(\frac{n(2n+1)}{6} + \frac{3n}{6}\right) \\
&= \frac{n+1}{2}\frac{2n^2+4n}{6} \\
&= \frac{n+1}{1}\frac{n^2+2n}{6} \\
&= \frac{n+1}{1}\frac{n(n+2)}{6} \\
&= \frac{n(n+1)(n+2)}{6}\ .
\end{aligned}
$$