Math Problem of the Month: November 2007

A ship leaves Oahu and sails 3,000 miles west. The captain then decides he needs to get back to Oahu as quickly as possible. What is the shortest distance he can sail back to Oahu? (Hint: The answer is not 3,000!)

Solution There are two types of circles on the globe (i.e. a sphere). Circles whose centers are the center of the sphere are called *great circles*. These include the equator, lines of longitude, and just about all navigational routes. This is because the shortest distance between two points is a "straight line", and a "strait line" (=geodesic) on the globe is a great circle.

Consider any line of latitude except the equator. This circle has a smaller radius and circumference, than the equator. How small? Well consider Oahu which lies at a latitude of 21.43° north. The radius of the earth is about $R = 3,965$ miles. Therefore, the radius of that line of latitude is $3,965 \cos 21.43° \approx 3,691$ miles $= r$. (See Figure 1.) So if a ship sails 3,000 miles west, then it sails 3,000 miles along this circle, intercepting an angle of (see Figure 2)

$$\theta = 3,000/3,691 \approx 0.813 \text{ radians} \approx 46.57° \ . \tag{1}$$

A chord joining these two endpoints (see Figure 3) has length

$$d = 2\sin(46.57°/2) \cdot 3,691 \approx 2,918 \text{ miles} \ . \tag{2}$$

This chord also lies on a great circle (because any two points determine a "line"). To figure out how far the ship is away from Oahu, we must reverse these calculations using a great circle. Starting with a chord of length $d = 2,918$ miles, we find the intercepted angle of a circle of radius 3,965 miles. We use equation (2) with the 46.57 as the unknown and 3,965 in place of 3,691. The angle is approximately $43.18° \approx 0.7537$ radians. We next use equation (1) with 3,965 in place if 3,691; 0.7537 in place of 0.813; and 3,000 as the unknown. **The ship is approximately $s = 2,988.4$ miles from Oahu.** This is not roundoff error. The difference is slight because of Oahu's proximity to the equator. Given the curvature of a circle (line of longitude) the disparity is greater when navigating (flying) over the continental United States.

These principles are routinely used in global navigation.

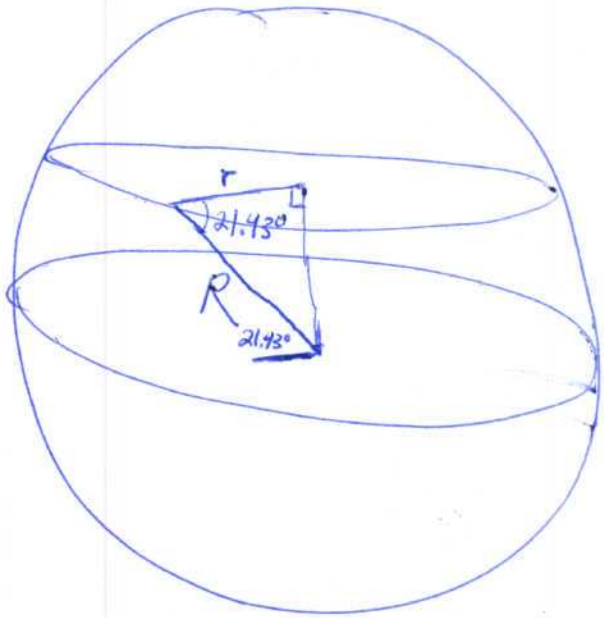

Fig 1

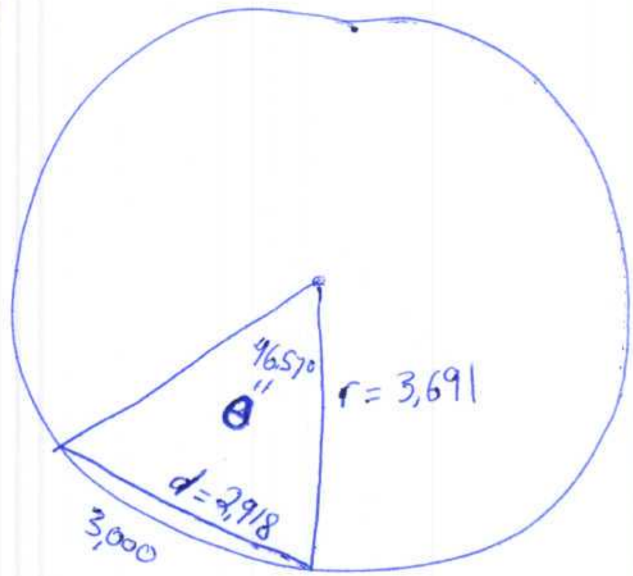

Fig 2

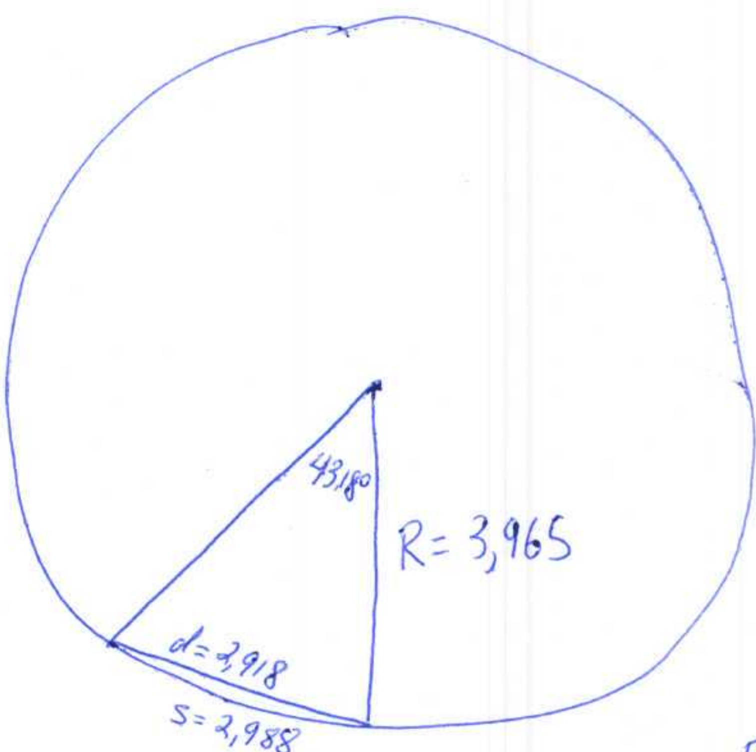

Fig 3